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Unmodeled Dynamics in Robust Nonlinear Control

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by

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Murat Arcak, Andrew Teel and Petar Kokotović. Robust nonlinear control of feedforward systems with unmodeled dynamics. To appear in *Automatica*.

Murat Arcak and Petar Kokotović. Robust nonlinear control of systems with input unmodeled dynamics. To appear in *Systems & Control Letters*.

Murat Arcak and Petar Kokotović. Nonlinear observers: A circle criterion design. In *Proceedings of the 38th IEEE Conference on Decision and Control*, pages 4872-4876, Phoenix, Arizona, December 1999.

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Abstract

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Since it is common to employ reduced models for control design, robustness to unmodeled dynamics is a crucial design criterion. Recent advances in nonlinear control theory have led to a number of recursive design procedures for which applications and extensions are being reported at an increasing rate. However, the robustness of these designs in the presence of unmodeled dynamics has received very little attention.

The purpose of this dissertation is to develop systematic redesign procedures that render nonlinear control laws robust against unmodeled dynamics. We consider classes of unmodeled dynamics characterized by their structural properties such as *input-to-state stability*, *passivity*, *minimum phaseness*, *relative degree*, and discuss their destabilizing effects on closed-loop stability. Using recently developed nonlinear feedback tools such as *nonlinear small-gain theorems* and *feedback passivation*, we develop redesign methods for each class of unmodeled dynamics considered.

Part One of the dissertation presents robust redesigns under the assumption that the full state of the plant is available for measurement. Our redesigns start with nominal control laws such as those designed by *backstepping* and *forwarding*, and robustify them to achieve *global asymptotic stability* in the presence of unmodeled dynamics.

Part Two addresses output-feedback design issues and presents a new nonlinear observer design. Compared to other areas of nonlinear control theory, progress in nonlinear output-feedback design has been slower due to the absence of constructive observer design methods. For systems with monotonic nonlinearities, we introduce a new global observer design which results in a nonlinear observer error system represented as the feedback interconnection of a linear system and a time-varying multivariable sector nonlinearity. Using efficient numerical methods available for linear matrix inequalities, observer gain matrices are computed to satisfy the circle criterion and, hence, to drive the observer error to zero.

Due to the absence of a *separation principle* for nonlinear systems, the availability of an observer does not mean that it can be used for feedback control. We discuss how the new observer can be incorporated in output-feedback design, and propose a small-gain method for output-feedback control design with robustness against unmodeled dynamics. The design is illustrated on the jet engine compressor example.

Contents

Chapter 1

Introduction

Nonlinear control theory has undergone a period of significant progress in the last decade. The emergence of new analysis tools such as *input-to-state stability*, nonlinear *small-gain* theorems, and the idea of rendering a system *passive* by feedback have lead to systematic design procedures such as *backstepping* and *forwarding*. Although such design methods are specialized for certain classes of systems, their extensions and combined use enlarge their range of applicability. Nonlinear control methods have already been successfully applied to the control of electrical motors, diesel engines, ships, jet engine compressors, and are promising for emerging technologies such as microelectromechanical systems.

The increasing demand for nonlinear control makes it necessary to improve the practicality of design methods by studying the effects of system uncertainty, disturbances, incomplete and noisy state measurements. While efforts in this direction have been successful in several specific problems including robustness against disturbances and parametric uncertainties, progress in other areas has been slower.

The main purpose of this dissertation is to study the robustness of nonlinear control methods in the presence of unmodeled dynamics. Because it is common to employ low order models for control design, robustness to unmodeled dynamics is a crucial design criterion. In general, a control design based on a *nominal* model fails to achieve stabilization in the presence of unmodeled dynamics. We study classes of nonlinear control laws such as those designed by backstepping and forwarding, and develop systematic redesign procedures for their robustification against unmodeled dynamics.

A common class of unmodeled dynamics are those that appear at the plant input. Among this class, two types of unmodeled dynamics affect closed-loop stability properties in fundamentally different ways and, hence, call for different redesign strategies. The first type is relative degree zero and minimum phase unmodeled dynamics. The second type consists of unmodeled dynamics that fail to meet at least one of the relative degree zero and minimum phase conditions and, hence, exhibit phase-lag at high frequencies. Although such "phase-lag" unmodeled dynamics are common in actuators, most results in the literature only address relative degree zero and minimum phase unmodeled dynamics. In this work we take a more ambitious path and study both types.

The second part of the dissertation addresses problems in which only the plant output is measured. Progress in nonlinear output-feedback control is hindered by two obstacles. First, nonlinear observers are available only for very restrictive classes of systems. Next, the availability of an observer does not imply that it can be used for output-feedback control, because the *separation principle* does not hold. In this work we introduce new tools for both observer design and observer-based control design.

For systems with monotonic nonlinearities, we introduce a new global observer design which results in a nonlinear observer error system represented as the feedback interconnection of a linear system and a time-varying multivariable sector nonlinearity. Using *linear matrix inequality* (LMI) software, observer gain matrices are computed to satisfy the circle criterion and, hence, to drive the observer error to zero. We discuss how the new observer can be incorporated in output-feedback design, and propose a small-gain method for output-feedback control design with robustness against unmodeled dynamics. The design is illustrated on the jet engine compressor example.

As in most nonlinear designs, our results are applicable to classes of systems characterized by their structural properties and types of nonlinearities. In contrast to a single design methodology that encompasses all nonlinear systems of interest, our approach offers the advantage of exploiting structural properties, and avoiding conservative results. In this work we emphasize global designs, that is, designs for the entire domain in which the system model is valid.

In Section 1.1 below, we review preliminary concepts that will be used throughout the dissertation. Section 1.2 presents a preview of the main topics discussed in the dissertation. Section 1.3 contains a list of notation and acronyms used in the dissertation.

1.1 Nonlinear Feedback Concepts

1.1.1 ISS Small-Gain Theorem

For systems with disturbances, a classical *total stability* concept is due to Malkin [60], Krasovskii [49], and Hahn [23]. Sontag [86] replaced this concept with a more useful concept of *input-to-state stability* (ISS). The system

$$\dot{x} = f(x, w), \quad f(0,0) = 0, \tag{1.1}$$

where w is a disturbance input, is ISS if there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that

$$|x(t)| \leq \max\left\{\beta(|x(0)|, t), \gamma\left(\sup_{0\leq\tau\leq t} |w(\tau)|\right)\right\}. \tag{1.2}$$

When the effect of the initial condition β vanishes as $t \to \infty$, the remaining term $\gamma(\cdot)$ is a class-$\mathcal{K}$ *ISS-gain* of the system (1.1).

Sontag and Wang [88] showed that the ISS property is equivalent to the existence of a positive definite and proper *ISS-Lyapunov function* $V(x)$ such that

$$|x| \geq \rho(|w|) \;\Rightarrow\; L_f V(x, w) \leq -\sigma(|x|), \tag{1.3}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ are class-$\mathcal{K}$ functions. With this $\rho(\cdot)$, the ISS-gain $\gamma(\cdot)$ in (1.2) is the composition $\gamma(\cdot) = \sigma_1^{-1} \circ \sigma_2 \circ \rho(\cdot)$, where

$$\sigma_1(|x|) \leq V(x) \leq \sigma_2(|x|). \tag{1.4}$$

Further characterizations of the ISS property are presented by Sontag and Wang [89].

The concept of ISS gain led to a new version of the ISS small-gain theorem by Jiang *et al.* [39], which includes the effect of initial conditions and represents an extension of an earlier theorem by Mareels and Hill [61]. It is now illustrated on the interconnected subsystems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_2, x_1). \end{aligned} \tag{1.5}$$

If the x_1-subsystem with x_2 as its input has ISS-gain $\gamma_1(\cdot)$, and the x_2-subsystem with x_1 as its input has ISS-gain $\gamma_2(\cdot)$, then the interconnection is globally asymptotically stable (GAS) if, for all $s > 0$,

$$\gamma_1 \circ \gamma_2(s) < s, \tag{1.6}$$

which is the ISS small-gain condition. Significant extensions and design applications of the ISS small-gain theorem are given by Teel [93].

1.1.2 Nonlinear Relative Degree and Zero Dynamics

The development of nonlinear geometric methods was a remarkable achievement of the 1980's, presented in the books by Isidori [25], Nijmeijer [67], Marino [63] and in the numerous papers referenced therein. Here, we describe two geometric concepts that will be used throughout the dissertation: nonlinear *relative degree* and *zero dynamics*. These indispensable tools bring into focus the common input-output structure of linear and nonlinear systems.

For the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \,, \\ y &= h(x) + j(x)u \,, \quad x \in I\!\!R^n \,, u, y \in I\!\!R \,, \end{aligned} \tag{1.7}$$

the relative degree at a point $x^\star$ is zero if $j(x^\star) \neq 0$, it is one if $j(x^\star)$ is identically zero on a neighborhood of $x^\star$ and $L_g h := \frac{\partial h}{\partial x} g(x) \neq 0$ at $x^\star$. This is so because

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = L_f h + L_g h \, u \,, \tag{1.8}$$

so that, if $L_g h$ is nonzero, then the input $u(t)$ appears in the expression for the first derivative $\dot{y}(t)$ of the output $y(t)$. If $L_g h$ is zero, we can differentiate $\dot{y}$ once more and check whether u appears in the expression for $\ddot{y}(t)$, etc.

When the system (1.7) has relative degree one, its *input-output linearization* is performed with the feedback transformation

$$u = (L_g h)^{-1}(v - L_f h) \quad \Rightarrow \quad \dot{y} = v \,, \tag{1.9}$$

which cancels the nonlinearities in the $\dot{y}$-equation and converts it into $\dot{y} = v$. Selecting new state coordinates in which y is one of the states, the remaining $n-1$ equations with $y(t) \equiv 0$ and $v(t) \equiv 0$ constitute the *zero dynamics*, that is, nonlinear dynamics which remain when the output is kept at zero. If the relative degree is two, then the linear part of the system is $\ddot{y} = v$, the chain of two integrators. In this case the zero dynamics are described by the remaining $n-2$ equations $y(t) = \dot{y}(t) \equiv 0$ and $v(t) \equiv 0$.

In *minimum phase systems* the zero dynamics are asymptotically stable. In *weakly minimum phase systems* the zero dynamics are stable, but not asymptotically stable.

1.1.3 Passivity and Feedback Passivation

Passivity, as a feedback concept, was first used by Popov [69, 70] in his frequency domain solution to the absolute stability problem. Popov's contribution led to various versions of the *circle criterion* by Sandberg [81], Narendra and Goldwyn [66], Zames [104], and to the *Positive Real* (PR) *Lemma* by Yakubovich [102] and Kalman [40]. For a minimal realization (A, B, C) of $H(s)$, the PR Lemma shows that $ReH(j\omega) \geq 0$ is equivalent to the existence of a $P = P^T > 0$ such that

$$A^T P + PA \leq 0 \tag{1.10}$$

$$B^T P = C. \tag{1.11}$$

Thus, $H(s)$ being passive (positive real) means that P satisfies not only the Lyapunov inequality (1.10), but also the input-output constraint (1.11) which restricts the relative degree of $H(s)$ to be less than two and its zeros to be stable. $H(s)$ is called *strictly positive real* (SPR) if (1.10) holds with strict inequality.

For a general nonlinear system H with state x, input u and output y, Willems [101] introduced a *storage function* $S(x)$, $S(0) = 0$, and a *supply rate* $w(u, y)$, and defined H as *dissipative* if $\dot{S}(x(t)) \leq w(u(t), y(t))$. Passivity is the special case when $w = u^T y$. If $H(s)$ is passive, then its storage function is obtained from the PR Lemma, that is, $S(x) = \frac{1}{2}x^T Px$, where P is as in (1.10)-(1.11).

The use of passivity as a nonlinear stabilization tool was made possible by the *Passivity Theorem* of Popov [71] and Zames [104], which states that the feedback interconnection of two nonlinear passive blocks H_1 and H_2 in Figure 1.1 is stable. This theorem encompasses the *circle criterion* where $H_1 = H_1(s)$ is positive real and H_2 is a time-varying nonlinearity $y_2 = \phi(t, u_2)$, which is passive because of its *sector* property $u_2\phi(t, u_2) \geq 0$.

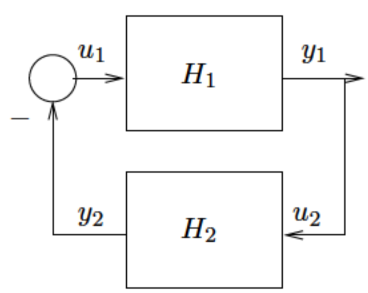

Figure 1.1: Passivity.

While passivity was a common tool in adaptive control, its first use for nonlinear control design was motivated by a difficulty encountered in feedback stabilization of linear-nonlinear cascade systems

$$\begin{aligned}\dot{x} &= f(x,\xi)\\ \dot{\xi} &= A\xi + Bu\,,\end{aligned}\tag{1.12}$$

resulting from input-output linearization. The difficulty was that GAS of the subsystem $\dot{x} = f(x,0)$ is not sufficient to achieve GAS of the whole cascade with ξ-feedback $u = K\xi$ alone. Thus, we need a feedback from both ξ and x, that is,

$$u = K\xi + v(x,\xi)\,.\tag{1.13}$$

Such a control law was designed by Byrnes and Isidori [9] for the special case of (1.12) with $\dot{\xi} = Bu$, where B is a square nonsingular matrix. Kokotović and Sussmann [47] extended this design to *feedback passivation* where the main idea is to make the cascade (1.12) appear as the feedback interconnection of the blocks H_1 and H_2 in Figure 1.1. The final result in Figure 1.2 is arrived at in several steps. First, an output η of the linear block H_1 is selected to be the input of the nonlinear block H_2, that is, the x-subsystem of (1.12) is rewritten as

$$\dot{x} = f(x,0) + g(x,\xi)\eta\,,\tag{1.14}$$

where several choices of $\eta = C\xi$ may be available. An output y is then chosen to render (1.14) passive from η to y. If a Lyapunov function $V(x)$ is known for $\dot{x} = f(x,0)$ so that $L_fV < 0$ for all $x \neq 0$, then $y = L_gV^T$ renders (1.14) passive because

$$\dot{V} = L_fV + L_gV\eta \leq L_gV\eta = y^T\eta\,.\tag{1.15}$$

Finally, if the linear block H_1 is made PR by feedback $K\xi$, the passivity theorem will be satisfied by closing the loop with $-y$ as in Figure 1.2.

This means the nonlinear feedback term in the control law (1.13) is $v(x,\xi) = -y = -L_gV^T$. What remains to be done is to find K and $P > 0$ to satisfy the *Feedback* PR Lemma

$$\begin{aligned}(A+BK)^TP + P(A+BK) &\leq 0\,,\\ B^TP &= C\,.\end{aligned}\tag{1.16}$$

Kokotović and Sussmann [47] showed that an FPR solution exists if and only if the minimal representation (A,B,C) is *relative degree one* and *weakly minimum phase.* Saberi *et al.* [79] showed that the weak minimum phase property of (A,B,C) is necessary unless some other restriction is imposed on the nonlinear part. An analysis

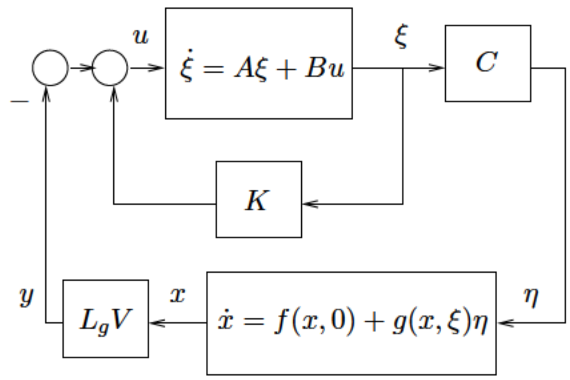

Figure 1.2: Feedback passivation design.

by Sepulchre [83], extending earlier results of Sussmann and Kokotović [91], revealed that the obstacle is the *peaking phenomenon*. Higher relative degree systems are prone to destabilizing transients caused by fast or slow peaking. For nonminimum phase systems global stabilization can be achieved only with further restrictions on the cross-term $g(x,\xi)$ as discussed by Braslavsky and Middleton [8], and Sepulchre and Arcak [84], where these restrictions are characterized by a relationship between the locations of the nonminimum phase zeros and the growth of $g(x,\xi)$ in x and ξ.

1.2 Preview of the Dissertation

1.2.1 Robustness against Unmodeled Dynamics

Control design methods like feedback linearization [25], backstepping [53], forwarding [93, 85] achieve GAS for classes of nonlinear systems. However, for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)p(\xi,x,u) \\ \dot{\xi} &= q(\xi,x,u)\,, \end{aligned} \tag{1.17}$$

where the ξ-subsystem with the output $v = p(x,\xi,u)$ represents unmodeled dynamics, a GAS control law $u = \alpha(x)$ designed for the nominal model $\dot{x} = f(x) + g(x)u$, will in general fail to achieve GAS of the actual system. Therefore, it is necessary to redesign control laws to make them robust against unmodeled dynamics. Several redesigns have been based on small-gain and passivity properties of the unmodeled dynamics.

As an illustration of small-gain redesigns proposed by Jiang *et al.* [73, 36, 39, 35, 34], Krstić *et al.* [54, 55], Praly and Wang [76], we let $p(\xi, x, u) = u + \xi$, $q(\xi, x, u) = q(\xi, x)$, and assume that the unmodeled dynamics are ISS with x considered as the input, that is,

$$|\xi(t)| \leq \max\left\{\beta_1(|\xi(0)|, t), \gamma_1\left(\sup_{0\leq\tau\leq t} |x(\tau)|\right)\right\}. \tag{1.18}$$

The nominal control law $\alpha(x)$ was designed for $V(x)$, such that $L_{f+g\alpha}V := \frac{\partial V}{\partial x}(f(x) + g(x)\alpha(x)) < 0$, $\forall x \neq 0$. We substitute $p(\xi, x, u) = u + \xi$ in (1.17)

$$\dot{x} = f(x) + g(x)[u + \xi], \tag{1.19}$$

and redesign the control law to assign an ISS-gain from ξ to x, that is,

$$|x(t)| \leq \max\left\{\beta_2(|x(0)|, t), \gamma_2\left(\sup_{0\leq\tau\leq t} |\xi(\tau)|\right)\right\}. \tag{1.20}$$

If $\gamma_2(\cdot)$ is selected such that that $\gamma_1 \circ \gamma_2(s) < s$, $\forall s \neq 0$, then the *ISS small-gain theorem* of Teel *et al.* [39, 93] guarantees GAS of the closed-loop system. The redesign of the control law is completed by a continuous approximation of the control law

$$u = \alpha(x) - sgn(L_gV(x))\rho(|x|), \tag{1.21}$$

where $\rho(\cdot)$ is determined from the desired ISS-gain $\gamma_2(\cdot)$ and the Lyapunov function $V(x)$. The resulting feedback system can tolerate all unmodeled dynamics that satisfy the ISS property (1.18), which represents its ISS-*gain margin*.

Small-gain redesigns for adaptive control were proposed by Jiang and Praly [37, 38], Jiang and Hill [33], and Jiang [32].

An alternative redesign by passivation does not require that unmodeled dynamics have bounded ISS-gain. Instead, the class of unmodeled dynamics is restricted by a passivity requirement on the ξ-subsystem in (1.17) with u as the input and $v = p(\xi, x, u)$ as the output.

The passivation redesigns of Janković *et al.* [31], extended by Hamzi and Praly [24], are based on $V(x)$ as a control Lyapunov function (CLF) for the nominal system $\dot{x} = f(x) + g(x)u$. For example, if $V(x)$ has the property

$$L_fV(x) < |L_gV(x)|^2, \quad \forall x \neq 0, \tag{1.22}$$

then the control law

$$u = -kL_gV(x), \quad k \geq 1 \tag{1.23}$$

guarantees GAS not only for the nominal system, but also for all stable unmodeled dynamics which satisfy the dissipativity condition

$$\dot{S}(\xi) \leq v\, u - \frac{1}{k} u^2, \tag{1.24}$$

where $S(\xi)$ is a storage function. Thus, (1.24) represents a 'stability margin' for the control law (1.23) because it defines a class of admissible unmodeled dynamics. This stability margin is guaranteed if, for example, the control law in (1.23) is optimal with the control penalty matrix $R(x) = I$ because, then, the value function $V(x)$ satisfies (1.22). For the case when $V(x)$ fails to satisfy (1.22), Janković *et al.* [31] construct a new $\tilde{V}(x)$ which recovers the same stability margin.

The first result in this dissertation, presented in Chapter 2, is a passivation redesign of backstepping control laws for

$$\begin{aligned} \dot{\chi} &= \Phi(\chi) + \Gamma(\chi) v & (1.25) \\ \dot{\xi} &= q(\xi, u) & (1.26) \\ v &= p(\xi, u)\,, & \end{aligned}$$

which robustifies them against unmodeled dynamics described by the ξ-subsystem (1.26). To preserve GAS in the presence of unmodeled dynamics satisfying (1.24), we want the CLF $\bar{V}(\chi)$ to be as in (1.22). It was shown by Hamzi and Praly [24] that (1.22) is equivalent to the existence of another CLF $V(\chi)$ such that

$$\limsup_{\chi \to 0} \frac{L_\Phi V(\chi)}{(L_\Gamma V(\chi))^2} \; < \; l, \tag{1.27}$$

for some $l > 0$. Then, using a positive scalar function $\theta(\cdot)$ such that

$$lim_{t\to\infty} \int_0^t \theta(s) ds = +\infty, \quad \theta(V(\chi)) > \frac{L_\Phi V(\chi)}{(L_\Gamma V(\chi))^2}\,, \quad \forall \chi \neq 0\,, \tag{1.28}$$

a redesigned $\bar{V}(\chi)$ which satisfies (1.22) is

$$\bar{V}(\chi) = \int_0^{V(\chi)} \theta(s) ds. \tag{1.29}$$

We derive conditions under which a CLF $V_n(\chi)$, constructed after n steps of backstepping, will satisfy the above condition. As an illustration, such a CLF is constructed for the system

$$\begin{aligned} \dot{X} &= X^3 + x \\ \dot{x} &= v \\ \dot{\xi} &= -\xi + \xi^3 u & (1.30) \\ v &= \xi^4 + u\,, \end{aligned}$$

with the unmodeled dynamics subsystem (1.30).

After two steps of backstepping we obtain the CLF

$$V_2 = \frac{1}{2}X^2 + \frac{1}{2}y^2\,, \quad y = x + X + X^3\,, \tag{1.31}$$

and using $\theta(V_2) = 3 + 18V_2^2$, we redesign the control law to be

$$u = -3y - \frac{9}{2}(X^2 + y^2)^2 y. \tag{1.32}$$

This control law achieves GAS for all unmodeled dynamics satisfying the dissipation inequality (1.24) with $k = 1$, including (1.30).

For linear unmodeled dynamics we present a version of this redesign which simplifies its applications to high order systems.

Another direction for achieving robustness against a wider class of unmodeled dynamics is the *dynamic nonlinear damping redesign* presented in Chapter 3. This redesign replaces passivity and small-gain restrictions by the less restrictive assumption that the unmodeled dynamics subsystem is relative degree zero and minimum phase. For the system (1.17), a nominal control law $\alpha(x)$ is redesigned to be

$$u = \alpha(x) - \kappa(1 + |m(t)| + |\alpha(x)|)L_gV(x), \quad \kappa > 0, \tag{1.33}$$

where the signal $|m(t)|$ constitutes an upper bound for the state of the unmodeled dynamics. We prove that this redesign guarantees GAS for sufficiently large κ.

To illustrate the dynamic nonlinear damping redesign, we consider the system

$$\dot{x} = x^2 + \Delta(s)u\,, \tag{1.34}$$

where the poles λ_i of the unmodeled dynamics $\Delta(s)$ satisfy $Re\{\lambda_i\} \leq -\delta < 0$. For the nominal control law $\alpha(x) = -x - x^2$, the redesigned control law is

$$\begin{aligned} u &= -x - x^2 - \kappa(1 + |m| + |\alpha|)x \\ \dot{m} &= -\delta m + |u|\,, \end{aligned}$$

which ensures that $|m(t)|$ is an upper bound for the state $\xi(t)$ of $\Delta(s)$.

The relative degree zero and minimum phase restrictions of the dynamic normalization redesign are tight, as illustrated by the simulation results in Figure 1.3. While the closed-loop system is GAS with the linear minimum phase unmodeled dynamics $\Delta_1(s)$, the nonminimum phase $\Delta_2(s)$ causes the closed-loop solutions to grow unbounded.

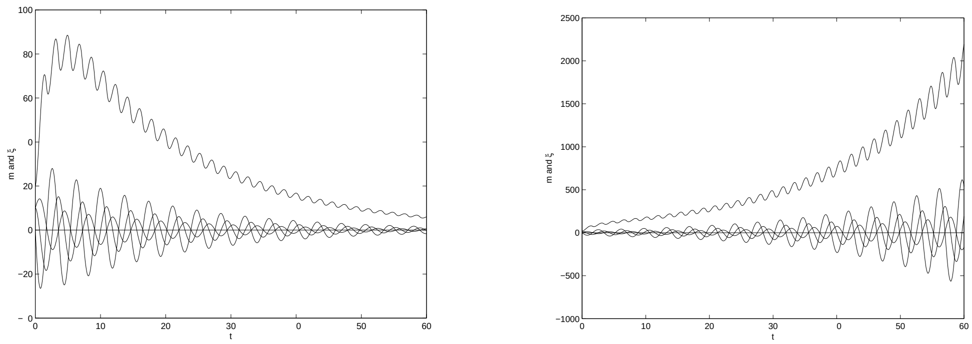

Figure 1.3: Closed-loop solutions for minimum phase $\Delta_1(s)$ (left), and nonminimum phase $\Delta_2(s)$ (right).

1.2.2 Phase-Lag Unmodeled Dynamics

The redesigns discussed so far require that the unmodeled dynamics be relative degree zero and minimum phase and, therefore, exclude several practically important classes of unmodeled dynamics such as those due to actuators. The high-gain nature of these redesigns results in reduced robustness to phase-lag unmodeled dynamics.

The problem of robustness against phase-lag unmodeled dynamics has received insufficient attention in the nonlinear control literature. A singular perturbation result by Sepulchre *et al.* [85, Theorem 3.18] shows that asymptotic stability can be preserved with large regions of attraction if the unmodeled dynamics are much faster than the nominal closed-loop system. For unmodeled dynamics with relative degree greater than zero, Praly and Jiang [74] have designed a semiglobal control law that incorporates a high-gain observer for the unmodeled dynamics. For the single integrator $\dot{x} = v$, Zhang and Ioannou [105] have considered nonminimum phase unmodeled dynamics and achieved global asymptotic stability using low-gain control laws.

In Chapter 4, we achieve global asymptotic stability for a broader class of systems with phase-lag unmodeled dynamics. We study systems in *feedforward* form, and redesign *nested saturation* control laws of Teel [92, 93].

We illustrate our result with a design for the system

$$\begin{aligned} \dot{x}_2 &= x_1 + x_1^2 + v^2 \\ \dot{x}_1 &= v + v^2 \,, \end{aligned} \tag{1.35}$$

and test its robustness against the nonminimum phase unmodeled dynamics

$$v(s) = \frac{-s+1}{s^2+s+1} u(s) \,. \tag{1.36}$$

For the nominal system, that is for (1.35) with $v = u$, the nested saturation design of Teel [92, 93] is applicable. With two saturation functions $\phi_i(s) = sgn(s)\min\{|s|, \lambda_i\}$ where $\lambda_1 = 0.5$, $\lambda_2 = 0.05$, and $y_1 = x_1$, $y_2 = x_1 + x_2$, the nested saturation control law is

$$u = -\phi_1(y_1 + \phi_2(y_2))\,. \tag{1.37}$$

This control law is not robust against the unmodeled dynamics (1.36): the solutions of the closed-loop system (1.35), (1.36), (1.37) grow unbounded as shown in Figure 1.4. We redesign the control law (1.37) to be

$$u = -\phi_1(k_1 x_1 + \phi_2(k_2 x_2))\,. \tag{1.38}$$

With parameters $\lambda_1 = 0.5$, $k_1 = 0.4$, $\lambda_2 = 0.05$, $k_2 = 0.02$, designed according to a small-gain procedure described in Chapter 4, the control law (1.38) renders the closed-loop system GAS, as illustrated in Figure 1.5.

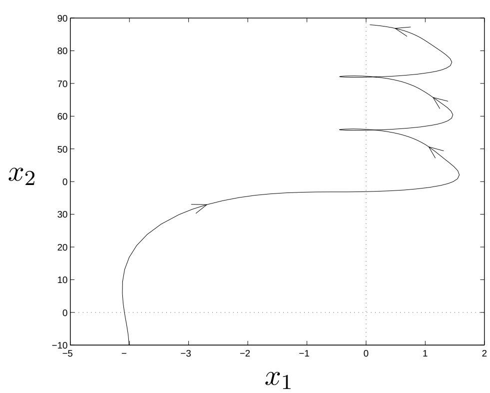

Figure 1.4: Nominal design.

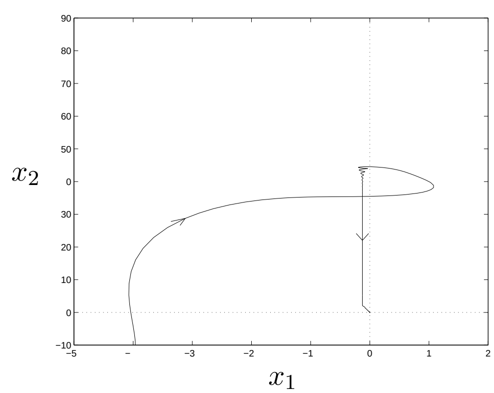

Figure 1.5: Robust redesign.

The achieved robustness property is due to the low-gain design (1.38), which is sufficient for stabilization of systems in feedforward form such as (1.35).

1.2.3 Output-Feedback Control

Compared with advances in other areas of nonlinear control theory, progress in nonlinear output-feedback design has been slower. First, nonlinear observers are available only for very restrictive classes of systems. Second, unlike linear systems where the *separation principle* allows output-feedback problems to be solved by combining state-feedback controllers with observers, the availability of a nonlinear observer does not imply that it can be used for output-feedback design because the *separation principle* does not hold.

Global nonlinear observer designs in the literature severely restrict classes of systems and nonlinearities. Early efforts by Thau [98], Kou *et al.* [48] and Banks [3] restricted the state-dependent nonlinearities to be globally Lipschitz. Under this restriction, quadratic Lyapunov functions have been used for observer design, with various extensions by Tsinias [99], Yaz [103], Boyd *et al.* [7, Section 7.6], Raghavan and Hedrick [77], Eker and Åström [14], and Rajamani [78].

For systems in which the nonlinearities appear as functions of the measured output, the observer design is linear because the nonlinearity is canceled by an "output injection" term. This class of systems has been characterized by Krener and Isidori [50], Bestle and Zeitz [6], Besançon [5], and other authors. *Output-injection observers* have been incorporated in observer-based control designs by Marino and Tomei [62, 63], Kanellakopoulos *et al.* [41, 53], Praly and Jiang [73], and, for stochastic nonlinear systems, by Deng and Krstić [13, 51], and Arslan and Başar [1].

A broader class of systems is characterized by linear dependence on unmeasured states. For this class, dynamic output-feedback designs have been proposed by Praly [72], Pomet *et al.* [68], Marino and Tomei [63], Freeman and Kokotović [19], and Praly and Kanellakopoulos [75].

The possibility to dominate the state dependent nonlinearities by linear high-gain has recently been explored by Khalil and coworkers [15, 46]. While $-kx$ cannot dominate x^3 globally, it can do so for as large $|x|$ as desired, provided the gain k is sufficiently large. To achieve this kind of "semiglobal" convergence, one must avoid the destabilizing effect of the *peaking phenomenon*, analyzed by Sussmann and Kokotović [91]. In Khalil's high-gain observer, this is achieved with saturation of the observer signals before they are fed to the controller. The high-gain observer has been employed in semiglobal output-feedback designs by Teel and Praly [96], Lin and Saberi [57], Lin and Qian [56], Praly and Jiang [74], and Isidori *et al.* [28]. The same approach has been employed in adaptive control by Janković [29], Khalil [43], and in the nonlinear *servomechanism* problem by Khalil *et al.* [42, 59, 45], and Isidori [26].

The idea of using a high-gain observer with saturation has led to the *semiglobal separation theorem* of Teel and Praly [95], which states that global stabilizability by state-feedback and *uniform observability* in the sense of Gauthier and Bornard [21] imply semiglobal stabilizability by dynamic output feedback. Several extensions and interpretations of this result have been presented by Atassi and Khalil [2], Isidori [27, Section 12.3], and Battilotti [4].

Global high-gain observers have been designed by Gauthier *et al.* [20] under a global Lipschitz condition - a common restriction in most global designs. In the absence of such a restriction, global stabilization by output feedback may not be possible, as shown by the counterexamples of Mazenc *et al.* [64].

1.2.4 A New Nonlinear Observer

In Chapter 5 we present an advance in observer design for systems with state-dependent nonlinearities. In the new observer, global convergence is achieved without high-gain. This advance is made under two restrictions which allow the observer error system to satisfy the well known *multivariable circle criterion*. First, a linear matrix inequality (LMI) is to be feasible, which implies a positive real property for the linear part of the observer error system. The second restriction is that the nonlinearities be nondecreasing functions of linear combinations of unmeasured states. This restriction ensures that the vector time-varying nonlinearity in the observer error system satisfies the sector condition of the circle criterion.

We illustrate the nonlinear observer design on the well known van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -x_1 + x_2 - \frac{1}{3}x_2^3\,. \end{aligned} \tag{1.39}$$

Our problem is to estimate x_2 when only x_1 is measured. The system nonlinearity x_2^3 which violates the global Lipschitz assumption depends on the unmeasured state. The necessary output injection conditions of Krener and Isidori [50] are not satisfied and, hence, none of the existing global observer design methods is applicable.

Our idea is to add a nonlinear injection term w^3 in the observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(\hat{x}_1 - x_1)\,, \\ \dot{\hat{x}}_2 &= -x_1 + \hat{x}_2 + l_2(\hat{x}_1 - x_1) - \frac{1}{3}w^3\,, \end{aligned} \tag{1.40}$$

so that the observer error $e := x - \hat{x}$ satisfies

$$\begin{aligned} \dot{e}_1 &= l_1 e_1 + e_2\,, \\ \dot{e}_2 &= l_2 e_1 + e_2 - \frac{1}{3}(x_2^3 - w^3)\,. \end{aligned} \tag{1.41}$$

The task of the nonlinear injection term w^3 is to counteract x_2^3 and achieve convergence of $e(t)$ to zero. For w, we are free to construct any function from the available signals x_1, $\hat{x}_1$, $\hat{x}_2$. For this construction, we use the Lyapunov function $V = e^T P e$. Its derivative along the observer error system (1.41) is

$$\dot{V} = e^T \left(P \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix} + \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix}^T P \right) e - \frac{2}{3} e^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} (x_2^3 - w^3) \,. \tag{1.42}$$

To render its quadratic part negative definite, we select P, l_1, l_2 such that

$$P \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix} + \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix}^T P + \nu I \,\leq\, 0 \,, \tag{1.43}$$

for some positive constant ν. Then, we design w to render the remainder of (1.42) nonpositive. Denoting

$$P = \begin{bmatrix} p & k \\ k & m \end{bmatrix} , \tag{1.44}$$

we see that w must guarantee $(ke_1 + me_2)(x_2^3 - w^3) \geq 0$. To this end we employ the inequality

$$(x_2 - w)(x_2^3 - w^3) \geq 0 \qquad \forall x_2, w \in I\!R \,, \tag{1.45}$$

which holds because x_2^3 is a nondecreasing function of x_2. In view of (1.45), we construct w to satisfy

$$ke_1 + me_2 = x_2 - w, \tag{1.46}$$

so that

$$w = x_2 - me_2 - ke_1 = (1-m)x_2 + m\hat{x}_2 + k(\hat{x}_1 - x_1) \,. \tag{1.47}$$

Since x_2 is not available, we select $m = 1$ and obtain

$$w = \hat{x}_2 + k(\hat{x}_1 - x_1) \,, \tag{1.48}$$

which, substituted in the observer (1.40), yields $\dot{V} < -\nu |e|^2$. This guarantees that $e(t) \to 0$ exponentially, provided (1.43) can be satisfied with $P > 0$, constrained by $m = 1$. We express this constraint as

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p & k \\ k & m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix} , \tag{1.49}$$

and combine it with (1.43) in the following matrix inequality:

$$\begin{bmatrix} P\begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix} + \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix}^T P + \nu I & P\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix} \\ \left(P\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix}\right)^T & 0 \end{bmatrix} \leq 0. \tag{1.50}$$

This inequality is linear in P, $P\,[l_1\ l_2]^T$, ν and k. Its feasibility can be determined numerically using the efficient LMI methods (see *e.g.* Boyd *et al.* [7]). If we set $k = 0$ in (1.48), that is, if we let $w = \hat{x}_2$, then the LMI is not feasible. With $k \neq 0$, the LMI is feasible and a solution is

$$P = \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}, \quad \nu = 2, \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -16 \end{bmatrix}, \quad k = -2\,. \tag{1.51}$$

This shows that it is crucial for the nonlinear injection w to contain the output error $k(\hat{x}_1 - x_1)$. The resulting nonlinear observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - 6(\hat{x}_1 - x_1) \\ \dot{\hat{x}}_2 &= -x_1 + \hat{x}_2 - 16(\hat{x}_1 - x_1) - \frac{1}{3}(\hat{x}_2 - 2(\hat{x}_1 - x_1))^3\,, \end{aligned} \tag{1.52}$$

guarantees global convergence $\hat{x}(t) \to x(t)$.

The constraint $m = 1$ imposed on P reveals that the nondecreasing property of the nonlinearity x_2^3 is not the only precondition for a successful design. The other required property is revealed by rewriting the observer error system (1.41) as

$$\dot{e} = \begin{bmatrix} l_1 & 1 \\ l_2 & 1 \end{bmatrix} e + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \vartheta\,, \quad \vartheta := -\frac{1}{3}(x_2^3 - w^3)\,, \tag{1.53}$$

and considering

$$z := [\,k \quad 1\,]\,e = x_2 - w \tag{1.54}$$

as the output of the linear block. Then, (1.53)-(1.54) is represented by the block-diagram in Figure 1.6. The key observation now is that the nondecreasing property (1.45) implies that $\varphi(t,z) := \frac{1}{3}(x_2^3 - w^3)$ is a sector nonlinearity: $z\varphi(t,z) \geq 0$. This provides a link with the well known circle criterion (see *e.g.* Khalil [44]), which guarantees $e(t) \to 0$ if the linear block is SPR. Indeed, (1.43) and (1.49) constitute the required SPR condition.

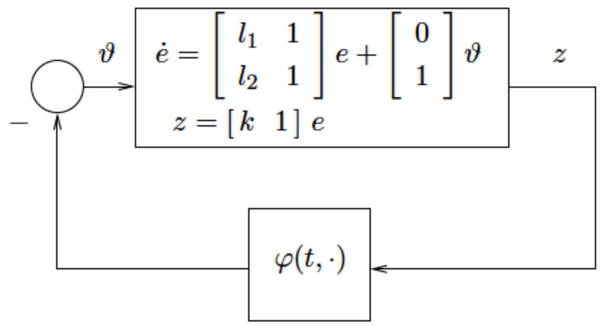

Figure 1.6: Observer error system.

In Chapter 5 we generalize this observer design to the control system

$$\begin{aligned} \dot{x} &= Ax + G\gamma(Hx) + \varrho(y, u) \\ y &= Cx\,, \end{aligned} \tag{1.55}$$

in which the state dependent nonlinearity $\gamma(Hx)$ is a vector where each entry is a nondecreasing function of a linear combination of the states. We also develop a reduced-order variant of this observer, which provides estimates only for the unmeasured states. Both full-order and reduced-order designs are constructive in the sense that the issues of existence and the evaluation of observer matrices are resolved by efficient LMI computations.

A further advantage of this observer is its robustness against inexact modeling of nonlinearities. We give bounds within which the observer error gradually increases with an increase in the modeling error.

In Chapter 6 we present structural conditions that characterize the feasibility of the new observer design. We derive necessary and sufficient conditions for feasibility of the full-order observer design, and prove that they are equivalent to the feasibility conditions of the reduced-order observer design.

1.2.5 Robust Output-Feedback Design

Due to the absence of a separation principle, the availability of a nonlinear observer does not imply that it can be used for output-feedback design. It was shown in Krstić *et al.* [53, Section 7.1] that, even with exponentially converging state estimates, a *certainty-equivalence* control law may lead to finite escape times.

Thus, nonlinear observers can only be used in conjunction with control laws that guarantee boundedness of the states in the presence of bounded observer errors. For classes of systems, such control laws have been designed by Freeman and Kokotović [17, 18], Krstić *et al.* [53], and Marino and Tomei [63].

In Chapter 7, the new observer is incorporated in output-feedback design in conjunction with the *observer backstepping* design of Krstić *et al.* [53]. In contrast to full state-feedback backstepping, this design employs the observer states as the *virtual control* variables, and introduces *nonlinear damping* terms to counteract the destabilizing effect of the observer error.

Finally, we use our observer-based control design tools to develop a robust output-feedback scheme for systems with unmodeled dynamics characterized by an ISS property. The resulting closed-loop system is depicted in Figure 1.7, where the unmodeled dynamics and the observer error system blocks have finite ISS gains. The remaining task for the control design is to assign the gains for the plant with respect to both observer error and unmodeled dynamics inputs so as to render the gains of both inner and outer loops less than one. Then, GAS is guaranteed via the ISS small-gain theorem of Teel *et al.* [39, 93].

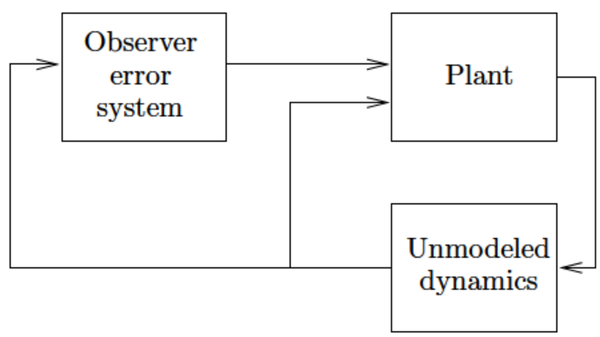

Figure 1.7: Closed-loop system with observer feedback.

In Chapter 7 we illustrate the design on an axial compressor model due to Moore and Greitzer [65], which has been the starting point for numerous jet engine control studies. We study the following single-mode approximation of the Moore-Greitzer

PDE model

$$\begin{aligned}
\dot{\phi} &= -\psi + \frac{3}{2}\phi + \frac{1}{2} - \frac{1}{2}(\phi+1)^3 - 3(\phi+1)R \\
\dot{\psi} &= \frac{1}{\beta^2}(\phi + 1 - u) \\
\dot{R} &= \sigma R(-2\phi - \phi^2 - R)\,, \quad R(0) \geq 0\,,
\end{aligned}$$

where ϕ and ψ are the deviations of the mass flow and the pressure rise from their set points, the control input u is the flow through the throttle, and, σ and β are positive constants.

Krstić *et al.* designed a state feedback GAS control law in [53, Section 2.4], and later replaced it by a design using ϕ and ψ in [52]. With a high-gain observer, Isidori [27, Section 12.7], and Maggiore and Passino [58], obtained a semiglobal result using the measurement of ψ alone. With $y = \psi$, our design achieves GAS. The exact observer cannot be designed because of the nonlinearities ϕR and $\phi^2 R$. However, the (ϕ, ψ)-subsystem (7.31),(7.32) contains the nondecreasing nonlinearity $(\phi+1)^3$, and we exploit this fact by treating the R-subsystem as unmodeled dynamics. The small-gain assignment is then achieved via an observer-backstepping design.

1.3 Notation and Terminology

A function $f : I\!R^n \to I\!R^q$ is C^k if its partial derivatives exist and are continuous up to order k, $1 \leq k < \infty$. A C^0 function is continuous. A C^∞ function is *smooth*, that is, it has continuous partial derivatives of any order. The same notation is used for vector fields.

A function $\sigma : I\!R_{\geq 0} \to I\!R_{\geq 0}$ is said to be class-$\mathcal{K}$ if it is continuous, increasing, and $\sigma(0) = 0$. It is called class-$\mathcal{K}_\infty$ if, in addition, $lim_{t\to+\infty}\sigma(t) = +\infty$.

A function $\beta : I\!R_{\geq 0} \times I\!R_{\geq 0} \to I\!R_{\geq 0}$ is said to be class-$\mathcal{KL}$ if for each $t \in I\!R_{\geq 0}$, $\beta(\cdot, t)$ is class-$\mathcal{K}$ and for each $s \in I\!R_{\geq 0}$, $\beta(s, \cdot)$ is decreasing and $lim_{t\to+\infty}\beta(s,t) = 0$.

A locally Lipschitz function $\phi(\cdot) : I\!R \to [-\lambda, \lambda]$ is called a *saturation function* with *saturation level* $\lambda > 0$, if $\phi(x) = x$ when $|x| \leq \lambda/2$, and

$$\lambda/2 \leq sgn(x)\phi(x) \leq \min\{|x|, \lambda\} \tag{1.56}$$

when $|x| \geq \lambda/2$. This definition incorporates the standard saturation function $\phi(x) = sgn(x)\min\{|x|, \lambda\}$.

Given a vector field $f : I\!R^n \to I\!R^n$ and a differentiable scalar function $\lambda : I\!R^n \to$

Table 1.1: List of acronyms.

CLF	control Lyapunov function	LES	local exponential stability
GAS	global asymptotic stability	LMI	linear matrix inequality
GES	global exponential stability	PR	positive real
IOS	input-to-output stability	SPR	strictly positive real
ISS	input-to-state stability		

$\mathbb{R}$, $L_f\lambda$ denotes the directional derivative of λ along f:

$$L_f\lambda(x) = \frac{\partial\lambda}{\partial x} f(x). \tag{1.57}$$

A smooth, positive definite and radially unbounded function $V(x)$ is called a *control Lyapunov function* (CLF) for the system $\dot{x} = f(x) + g(x)u$ if, for all $x \neq 0$,

$$L_gV(x) = 0 \quad \Rightarrow \quad L_fV(x) < 0. \tag{1.58}$$

Throughout the dissertation, $|\cdot|$ denotes the Euclidean norm. We say that a measurable function $u(t)$ is locally bounded if, for all $T > 0$, $\sup_{t\in[0,T]} |u(t)| < \infty$, where sup stands for the essential supremum. For a locally bounded $u(t)$, we define the $\mathcal{L}_\infty$ and *asymptotic norms* as

$$\|u\|_\infty := \sup_{t\geq 0} |u(t)|, \quad \|u\|_a := \limsup_{t\to\infty} |u(t)|, \tag{1.59}$$

respectively.

Table 1.1 contains a list of acronyms used throughout the dissertation.

Part I

Robust Redesigns Against Unmodeled Dynamics

Chapter 2

Robustification of Backstepping

In this chapter we redesign backstepping schemes such as those in Kanellakopoulos *et al.* [41], and Krstić *et al.* [53], to robustify them against input unmodeled dynamics. We consider systems of the form

$$\begin{aligned}
\dot{X} &= F(X) + G(X)x_1 \\
\dot{x}_1 &= f_1(X, x_1) + g_1(X, x_1)x_2 \\
\dot{x}_2 &= f_2(X, x_1, x_2) + g_2(X, x_1, x_2)x_3 \\
\vdots &= \vdots \\
\dot{x}_n &= f_n(X, x) + g_n(X, x)v
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\dot{\xi} &= q(\xi, u) \\
v &= p(\xi, u),
\end{aligned} \tag{2.2}$$

where $|g_i(X, ..., x_i)| \geq g_0 > 0$, $\forall (X, ..., x_i) \in I\!R^{r+i}$, $i = 1, \cdots, n$. The ξ-subsystem (2.2) with input $u \in I\!R$ and output $v \in I\!R$ represents unmodeled dynamics, that is, (2.1) with $v = u$ is the nominal system. When $u = 0$, the system (2.1),(2.2) has an equilibrium at zero, whose stability properties are to be analyzed.

For the X-subsystem with x_1 viewed as a virtual control input, a CLF $V_0(X)$ and a control law $\Lambda^0(X)$, $\Lambda^0(0) = 0$, are known such that, for all $X \neq 0$,

$$L_{F+G\Lambda^0} V_0(X) := \frac{\partial V_0}{\partial X}(F + G\Lambda^0) = -U_0(X) < 0\,. \tag{2.3}$$

With the knowledge of $V_0(X)$ and $\Lambda^0(X)$, backstepping can be applied to guarantee GAS for the nominal system. However, a GAS control law for the nominal system does not guarantee GAS in the presence of unmodeled dynamics. To robustify backstepping designs, we propose two *redesign* methods: *passivation* and *truncated passivation*. In the first redesign we use the results of Hamzi and Praly [24], and ensure GAS via the passivity properties of the closed-loop system. In the second redesign we passivate the X-subsystem, and proceed with backstepping. For both redesigns the unmodeled dynamics subsystem (2.2) is restricted to be minimum phase and relative degree zero.

In Section 2.1 we review the two main versions of backstepping: *cancelation backstepping* and *L_GV-backstepping*. In Sections 2.2 and 2.3, we present the two redesigns. $V_0(X)$, $\Lambda^0(X)$, and the system (2.1), (2.2) are assumed to be sufficiently smooth.

2.1 Cancelation and L_GV-Backstepping

Backstepping design starts with the virtual control law $\Lambda^0(X)$ designed for the X-subsystem as in (2.3). By adding $-\eta_0 L_G V_0(X)$ to $\Lambda^0(X)$ we obtain the virtual control law

$$\Lambda(X) = \Lambda^0(X) - \eta_0 L_G V_0(X), \qquad \eta_0 \geq 0, \tag{2.4}$$

which, when $\eta_0 > 0$, increases the negativity of $L_{F+G\Lambda}V_0(X)$. With the error variable $y_1 := x_1 - \Lambda(X)$, the (X, x_1)-subsystem driven by x_2 is

$$\begin{aligned} \dot{X} &= (F + G\Lambda)(X) + G(X)y_1 && (2.5)\\ \dot{y}_1 &= f_1(X, x_1) - \dot{\Lambda}(X, y_1) + g_1(X, x_1)x_2, && (2.6) \end{aligned}$$

where $\dot{\Lambda}(X, y_1)$ is explicitly known from (2.4) and (2.5).

Step 1. To find a virtual control law $\alpha_1^0(X, x_1)$ for the (X, x_1)-subsystem, we introduce the CLF

$$V_1(X, x_1) := V_0(X) + \frac{1}{2\mu_1} y_1^2, \tag{2.7}$$

where $\mu_1 > 0$ is to be specified. Its time derivative along (2.5) and (2.6) is

$$\begin{aligned} \dot{V}_1(X, x_1) \leq\ & -U_0(X) - \eta_0 (L_G V_0(X))^2 + L_G V_0(X) y_1 \\ & + \frac{1}{\mu_1} y_1 \Big(f_1(X, x_1) - \dot{\Lambda}(X, y_1) + g_1(X, x_1)x_2 \Big). \end{aligned} \tag{2.8}$$

Our aim is to select $x_2 = \alpha_1^0(X, x_1)$ that renders $\dot{V}_1$ negative definite. Using

$$-\eta_0 (L_G V_0(X))^2 + L_G V_0(X) y_1 \leq \frac{1}{4\eta_0} y_1^2, \tag{2.9}$$

we obtain

$$\dot{V}_1(X, x_1) \leq -U_0(X) + \frac{1}{4\eta_0} y_1^2 + \frac{1}{\mu_1} y_1 \Big(f_1(X, x_1) - \dot{\Lambda}(X, y_1) + g_1(X, x_1)x_2 \Big), \tag{2.10}$$

which suggest the virtual control law

$$\alpha_1^0(X,x_1) = \frac{1}{g_1(X,x_1)}\left[-k_1y_1 - f_1(X,x_1) + \dot{\Lambda}(X,y_1)\right], \quad k_1 > 0. \tag{2.11}$$

Then, $x_2 = \alpha_1^0(X,x_1)$ yields

$$\dot{V}_1(X,x_1) \le -U_0(X) + \left(\frac{1}{4\eta_0} - \frac{k_1}{\mu_1}\right)y_1^2. \tag{2.12}$$

Setting $0 < \mu_1 < 4k_1\eta_0$, we guarantee that $\dot{V}_1(X,x_1)$ is negative definite.

We refer to this design as *L_GV-backstepping* because the sign-indefinite term $L_GV_0(X)y_1$ in (2.8) is dominated by adding $-\eta_0 L_GV_0(X)$ to the previous virtual control law $\Lambda^0(X)$. In *cancelation backstepping* we keep the control law $\Lambda(X) = \Lambda^0(X)$ by setting $\eta_0 = 0$ in (2.4). In this case, the $L_GV_0(X)y_1$ term in (2.8) must be canceled by $\alpha_1^0(X,x_1)$. A virtual control law incorporating both types of backstepping is

$$\alpha_1^0(X,x_1) = \frac{1}{g_1(X,x_1)}[-k_1y_1 - \lambda_1 L_GV_0(X) - f_1(X,x_1) + \dot{\Lambda}(X,y_1)], \tag{2.13}$$

where, for cancelation backstepping, $\lambda_1 > 0$ and $\mu_1 = \lambda_1$ in (2.7). If $\eta_0 > 0$, we recover the L_GV-backstepping control law (2.11) with $\lambda_1 = 0$.

For the (X,y_1)-subsystem (2.5),(2.6) driven by x_2, the input vector field is given by $[0\,,\,g_1(X,x_1)]^T$. Differentiating (2.7) along this vector field, the counterpart of $L_GV_0(X)$ for the X-subsystem is $g_1(X,x_1)y_1$. Then, as in (2.4), we introduce the virtual control

$$\alpha_1(X,x_1) = \alpha_1^0(X,x_1) - \eta_1 g_1(X,x_1)y_1, \quad \eta_1 \ge 0, \tag{2.14}$$

which enables us to avoid a cancelation in the next step, and define the error variable $y_2 := x_2 - \alpha_1(X,x_1)$.

Step i. $(i = 2,\cdots,n)$ We take the CLF

$$V_i(X,x_1,...,x_i) = V_0(X) + \frac{1}{2\mu_1}y_1^2 + \ ... \ + \frac{1}{2\mu_i}y_i^2, \tag{2.15}$$

where $\mu_i > 0$ is to be specified. With $k_i > 0$, calculations similar to Step 1 yield the virtual control law

$$\begin{aligned}\alpha_i^0(X,...,x_i) \quad = \quad & \frac{1}{g_i(X,...,x_i)}[-k_iy_i - f_i(X,...,x_i) \\ & \quad +\dot{\alpha}_{i-1}(X,...,y_i) - \lambda_i g_{i-1}(X,...,x_{i-1})y_{i-1}],\end{aligned} \tag{2.16}$$

where $\lambda_i > 0$ in cancelation backstepping, while $\eta_{i-1} > 0$ and $\lambda_i = 0$ in $L_G V$-backstepping. By selecting $\mu_i = \lambda_i \mu_{i-1}$ in the first, and, $0 < \mu_i < 4k_i\eta_{i-1}$ in the second case, $\dot{V}_i(X, x_1, ..., x_i)$ is rendered negative definite. For $i < n$, we define

$$\alpha_i(X, ..., x_i) := \alpha_i^0(X, ..., x_i) - \eta_i g_i(X, ..., x_i) y_i, \ \ \eta_i \geq 0,$$

and the next error variable $y_{i+1} := x_{i+1} - \alpha_i(X, ..., x_i)$.

This procedure results in the CLF

$$V_n(X, x) = V_0(X) + \frac{1}{2\mu_1} y_1^2 + \ ... \ + \frac{1}{2\mu_n} y_n^2, \tag{2.17}$$

and the control law

$$\begin{aligned} u = \alpha_n(X, x) = & \frac{1}{g_n(X, x)}[-k_n y_n - f_n(X, x) \\ & + \dot{\alpha}_{n-1}(X, y_1, ..., y_{n-1}) - \lambda_n g_{n-1}(X, ..., x_{n-1}) y_{n-1}], \end{aligned} \tag{2.18}$$

which renders $\dot{V}_n(X, x_1, ..., x_n)$ negative definite, thus achieving GAS for the nominal system.

2.2 Passivation Redesign

In this section we consider *input strictly passive* unmodeled dynamics, characterized by a constant $\delta > 0$ and a positive definite, radially unbounded function $S(\xi)$ such that

$$\dot{S}(\xi) \leq -\delta u^2 + v\, u. \tag{2.19}$$

For linear unmodeled dynamics $v = \Delta(s)u$, this means

$$Re\{\Delta(j\omega)\} \geq \delta, \qquad \forall \omega \in I\!R. \tag{2.20}$$

In compact notation, the system (2.1),(2.2) is

$$\begin{aligned} \dot{\chi} &= \Phi(\chi) + \Gamma(\chi) v & (2.21) \\ \dot{\xi} &= q(\xi, u) & (2.22) \\ v &= p(\xi, u), \end{aligned}$$

where

$$\begin{aligned} \chi &:= [X^T, x^T]^T \\ \Phi(\chi) &:= [F(X) + G(X)x_1, \cdots, f_n(X, x)]^T \\ \Gamma(\chi) &:= [0, \cdots, g_n(X, x))]^T. \end{aligned}$$

The passivation redesign makes use of the following lemma:

Lemma 2.1 *Consider the system (2.21),(2.22), and suppose that the ξ-subsystem satisfies (2.19) with $\delta > 0$, and is GAS when $u = 0$. If there exists a positive definite and radially unbounded function $\bar{V}(\chi)$ such that, for all $\chi \in I\!R^{r+n} - \{0\}$,*

$$L_\Phi \bar{V}(\chi) < (L_\Gamma \bar{V}(\chi))^2, \tag{2.23}$$

then the control law

$$u = -kL_\Gamma \bar{V}(\chi), \qquad k \geq \frac{1}{\delta}, \tag{2.24}$$

guarantees GAS.

Proof: With the control law (2.24), we view the closed-loop system as the feedback interconnection of the subsystems (2.21) and (2.22), and examine the passivity properties of each subsystem. For (2.21), we use $k\bar{V}(\chi)$ as a storage function, and denote $\beta(\chi) := (L_\Gamma \bar{V}(\chi))^2 - L_\Phi \bar{V}(\chi) > 0$, $\forall \chi \neq 0$. Substituting (2.24), we obtain

$$k\dot{\bar{V}}(\chi) = \frac{1}{k}u^2 - uv - k\beta(\chi). \tag{2.25}$$

Adding (2.19) and (2.25), we get

$$k\dot{\bar{V}}(\chi) + \dot{S}(\xi) \leq -(\delta - \frac{1}{k})u^2 - k\beta(\chi) \leq -k\beta(\chi), \tag{2.26}$$

which establishes global stability of the closed-loop system (2.21),(2.22) with (2.24). It also follows from (2.26) that $\chi \to 0$. Since $u = 0$ when $\chi = 0$, and the ξ-subsystem is GAS, we conclude from LaSalle's invariance principle that $\xi \to 0$, and hence, the closed-loop system is GAS. □

The existence of a function satisfying (2.23) for general nonlinear systems has been shown by Hamzi and Praly [24] to be equivalent to the existence of a CLF $V(\chi)$ and a constant $l > 0$ such that

$$\limsup_{\chi \to 0} \frac{L_\Phi V(\chi)}{(L_\Gamma V(\chi))^2} < l. \tag{2.27}$$

Under this local condition, a continuous, positive definite function $\theta : I\!R_{\geq 0} \to I\!R_{\geq 0}$ exists such that

$$\theta(V(\chi)) > \frac{L_\Phi V(\chi)}{(L_\Gamma V(\chi))^2}, \quad \forall \chi \in I\!R^{r+n} - \{0\}, \tag{2.28}$$

$$lim_{t \to \infty} \int_0^t \theta(s) ds = +\infty. \tag{2.29}$$

Then, it can be verified that

$$\bar{V}(\chi) = \int_0^{V(\chi)} \theta(s)ds \tag{2.30}$$

is a positive definite and radially unbounded function which satisfies (2.23). It is therefore useful to determine when a CLF $V(\chi)$ will satisfy (2.27).

Lemma 2.2 *Let* $P := \left[\frac{\partial^2 V}{\partial \chi^2}\right]_{\chi=0}$. *If* $\frac{1}{2}\chi^T P \chi$ *is a CLF for the Jacobian linearization of (2.21), then* $V(\chi)$ *satisfies (2.27).* □

We now show when this will be the case for $V_n(\chi)$ constructed by backstepping.

Theorem 2.1 *Consider the system (2.21)-(2.22), and suppose that the* ξ*-subsystem satisfies (2.19) with* $\delta > 0$*, and is GAS when* $u = 0$*. If*

$$P_0 = \left[\frac{\partial^2 V_0}{\partial X^2}\right]_{X=0} \quad and \quad Q_0 = \left[\frac{\partial^2 U_0}{\partial X^2}\right]_{X=0} \tag{2.31}$$

in (2.3) are both positive definite, then there exists a positive definite function $\bar{V}(\chi)$ *such that the control law*

$$u = -kL_\Gamma \bar{V}(\chi), \quad k \geq \frac{1}{\delta}, \tag{2.32}$$

renders the closed-loop system GAS.

Proof: With the procedure of Section 2.1 applied to (2.21), the quadratic part of $V_n(\chi)$ in (2.17)

$$\frac{1}{2}X^T P_0 X + \frac{1}{2\mu_1}y_1^2 + \cdots + \frac{1}{2\mu_n}y_n^2$$

is a CLF for the Jacobian linearization of (2.21). It follows from Lemma 2.2 and the preceding discussion that there exists a positive definite and radially unbounded $\bar{V}(\chi) = \int_0^{V_n(\chi)} \theta(s)ds$ which satisfies (2.23). Then, by Lemma 2.1, (2.32) achieves GAS. □

Substituting $L_\Gamma \bar{V} = \theta(V_n)\frac{1}{\mu_n}g_n(X,x)y_n$ in (2.32), the redesigned control law is

$$u = \frac{-k}{\mu_n}\theta(V_n)g_n(X,x)y_n, \quad k \geq \frac{1}{\delta}, \tag{2.33}$$

where $\theta : I\!R_{\geq 0} \to I\!R_{\geq 0}$ is a continuous, positive definite function that satisfies (2.28) and (2.29). It is important to note that this redesign does not require detailed information about the unmodeled dynamics, only a lower bound on δ is assumed to be known.

Example 2.1 *Consider the system with linear unmodeled dynamics* $\Delta(s)$*:*

$$\begin{aligned}\dot{X} &= X^3 + x \\ \dot{x} &= v \\ v &= \frac{12(s+35)(s+20)}{7(s+40)(s+30)}u = \Delta(s)u,\end{aligned} \tag{2.34}$$

Since $Re\{\Delta(j\omega)\} \geq 1$*,* $\Delta(s)$ *satisfies (2.19) with* $\delta = 1$*. Using the virtual control* $\Lambda^0(X) = -X - X^3$ *for the* X*-subsystem, we see that (2.3) is satisfied with* $V_0(X) = \frac{1}{2}X^2$*,* $U_0(X) = X^2$*, that is,* $P_0 = 1$*,* $Q_0 = 2$ *in (2.31). We define the error variable* $y = x + X + X^3$ *and set* $\mu_1 = 1$ *in (2.17), that is,* $V_2(X, x) = \frac{1}{2}X^2 + \frac{1}{2}y^2$*. In the* (X, y)*-coordinates the system is*

$$\begin{aligned}\dot{X} &= -X + y \\ \dot{y} &= (1 + 3X^2)(-X + y) + v,\end{aligned} \tag{2.35}$$

and, hence,

$$\begin{aligned}L_\Phi V_2(X, y) &= -X^2 - 3X^3 y + (1 + 3X^2)y^2 \\ L_\Gamma V_2(X, y) &= y.\end{aligned} \tag{2.36}$$

We now need to find a continuous function $\theta(V_2)$ *such that*

$$\theta(V_2) > \frac{L_\Phi V_2(X, y)}{(L_\Gamma V_2(X, y))^2} = 1 + 3X^2 + X^2\left(\frac{-3X}{y} - \frac{1}{y^2}\right).$$

Using the inequality

$$\frac{r}{y} - \frac{1}{y^2} \leq \frac{r^2}{4}, \quad \forall y \in I\!R,$$

we obtain, upon completion of the squares,

$$\frac{L_\Phi V_2(X, y)}{(L_\Gamma V_2(X, y))^2} \leq 1 + 3X^2 + \frac{9}{4}X^4 < 3 + 18V_2(X, y)^2.$$

Choosing $\theta(s) = 3 + 18s^2$*,* $k = \frac{1}{\delta} = 1$*, and substituting in (2.33), we obtain the redesigned control law*

$$u = -3y - \frac{9}{2}(X^2 + y^2)^2 y, \tag{2.37}$$

which achieves GAS. $\square$

2.3 Truncated Passivation Redesign

For higher order systems the task of finding the function $\theta(\cdot)$ may be cumbersome. We now circumvent this difficulty by a 'truncated' design in which we passivate the X-subsystem only, and then apply backstepping to the redesigned virtual control law.

For the Jacobian linearization of the X-subsystem, $\frac{1}{2}X^T P_0 X$ is a CLF, provided P_0 and Q_0 defined in (2.31) are positive definite. Therefore, we conclude from Lemma 2.2 that there exists a positive definite and radially unbounded function $\bar{V}_0(X)$ that satisfies

$$L_F\bar{V}_0(X) < L_G\bar{V}_0(X)^2, \quad \forall X \in I\!R^r - \{0\}. \tag{2.38}$$

Then, the virtual control law

$$\Lambda(X) = -k_0 L_G\bar{V}_0(X), \qquad k_0 \geq 1, \tag{2.39}$$

renders $\dot{\bar{V}}_0(X) = L_F\bar{V}_0(X) + L_G\bar{V}_0(X)\Lambda(X)$ negative definite.

To make the main features of the truncated passivation redesign more apparent, we present it for the special case of (2.1) in which the X-subsystem is augmented by a chain of n integrators, and the unmodeled dynamics are linear:

$$\begin{array}{rcl} \dot{X} &=& F(X) + G(X)x_1 \\ \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& x_3 \\ \vdots &=& \vdots \\ \dot{x}_n &=& \Delta(s)u. \end{array} \tag{2.40}$$

For this subclass, the sequence of the virtual control laws obtained via L_GV-backstepping is

$$\begin{array}{rcl} \alpha_1 &=& -k_1(x_1 - \Lambda) + \dot{\Lambda} \\ \alpha_2 &=& -k_2(x_2 - \alpha_1) + \dot{\alpha}_1 \\ \vdots &=& \vdots \\ \alpha_n &=& -k_n(x_n - \alpha_{n-1}) + \dot{\alpha}_{n-1}, \end{array} \tag{2.41}$$

where $k_i > 0, \quad i = 1, 2, \cdots, n$, and Λ is as in (2.39).

To represent the closed-loop system as the feedback interconnection of a linear and a nonlinear subsystem, we write the $\dot{x}_n$-equation of (2.40) with $u = \alpha_n$ in the Laplace domain as

$$sx_n = \Delta(s)[-k_n(x_n - \alpha_{n-1}) + s\alpha_{n-1}], \tag{2.42}$$

which yields

$$x_n = \frac{(s+k_n)\Delta(s)}{s+k_n\Delta(s)}\alpha_{n-1} =: \tilde{\Delta}_1(s)\alpha_{n-1}. \tag{2.43}$$

Next, we substitute $x_n = sx_{n-1}$ and $\alpha_{n-1} = -k_{n-1}x_{n-1} + (s+k_{n-1})\alpha_{n-2}$ in (2.43), and obtain

$$x_{n-1} = \frac{(s+k_{n-1})\tilde{\Delta}_1(s)}{s+k_{n-1}\tilde{\Delta}_1(s)}\alpha_{n-2} =: \tilde{\Delta}_2(s)\alpha_{n-2}. \tag{2.44}$$

Proceeding recursively, we get

$$x_1 = \tilde{\Delta}_n(s)\Lambda(X), \tag{2.45}$$

where $\tilde{\Delta}_n(s)$ is obtained from

$$\begin{aligned} \tilde{\Delta}_0(s) &:= \Delta(s) \\ \tilde{\Delta}_i(s) &:= \frac{(s+k_{n-i+1})\,\tilde{\Delta}_{i-1}(s)}{s+k_{n-i+1}\tilde{\Delta}_{i-1}(s)}, \quad i = 1,\cdots,n. \end{aligned} \tag{2.46}$$

Thus, the closed-loop system (2.40)-(2.41) is

$$\begin{aligned} \dot{X} &= F(X) + G(X)x_1 \\ x_1 &= \tilde{\Delta}_n(s)\Lambda(X). \end{aligned} \tag{2.47}$$

Because $\Lambda(X)$ in (2.39) is in L_GV-form, we can use Lemma 2.1 to prove the following stability margin:

Lemma 2.3 *The closed-loop system (2.47) is GAS for all stable $\tilde{\Delta}_n(j\omega)$ that satisfy*

$$Re\{\tilde{\Delta}_n(j\omega)\} \geq \frac{1}{k_0}, \quad \forall\omega \in I\!R. \tag{2.48}$$

□

Our final result shows how the design parameters are to be selected to satisfy (2.48).

Theorem 2.2 *Consider the system (2.40), with a controller designed according to (2.39) and (2.41). Suppose that* $\Delta(s)$ *is minimum phase and relative degree zero, with high-frequency gain*

$$h := \lim_{\omega\to\infty} \Delta(j\omega) > 0.$$

If we select $k_0 > \max\{1, \frac{1}{h}\}$*, and* $k_i > 0,\ i = 1, \cdots, n-1$*, then there exists* $k^\star > 0$ *such that* $k_n \geq k^\star$ *guarantees GAS for the closed-loop system.*

Proof: From (2.46), $\tilde{\Delta}_{n-1}(s)$ is minimum phase, relative degree zero and its high-frequency gain is $h > 0$. This implies that

$$\tilde{\Delta}_n(s) = \frac{(s+k_1)\,\tilde{\Delta}_{n-1}(s)}{s + k_1\tilde{\Delta}_{n-1}(s)} \tag{2.49}$$

is stable for sufficiently large $k_1 > 0$, as verified from a root-locus argument. Next, it can be shown from (2.49) that, as k_1 is increased, the Nyquist plot of $\tilde{\Delta}_n(s)$ converges to that of $\frac{h(s+1)}{s+h}$, which is a circle that intersects the real axis at 1 and h. Since $\frac{1}{k_0} < \min\{1, h\}$, (2.48) is satisfied if k_1 is selected sufficiently large. Thus, GAS follows from Lemma 2.3. □

Example 2.2 *For the system of Example 2.1, we now perform the passivation redesign for the* X*-subsystem only, and then apply backstepping. A redesigned control law for the* X*-subsystem is* $\Lambda(X) = -2k_0X^3$*. Then, the* L_GV*-backstepping control law*

$$u = -k_1(x + 2k_0X^3) - 6k_0X^2(X^3 + x), \tag{2.50}$$

achieves GAS for the nominal model. To guarantee GAS with $\Delta(s)$*, we note that* $h = 1$*, and choose* $k_0 > 1$*. Then, by Theorem 2.2, the closed-loop system (2.34),(2.50) is GAS for sufficiently large* $k_1 > 0$*.* □

2.4 Summary

We have presented two passivation redesigns of backstepping which achieve global asymptotic stability for a class of minimum phase unmodeled dynamics with relative degree zero. The redesigns do not require detailed information about the unmodeled dynamics: the class of admissible unmodeled dynamics are characterized by their passivity properties in the first redesign, and their high-frequency gains in the second redesign. Our analysis provides insight into the robustness properties of backstepping designs.

Chapter 3

Dynamic Nonlinear Damping Redesign

The redesigns presented so far restrict the unmodeled dynamics by small-gain or passivity conditions. The *dynamic nonlinear damping* redesign presented in this chapter removes these restrictions. Instead, the main restriction is that the unmodeled dynamics subsystem be relative degree zero and minimum phase. For nonlinear unmodeled dynamics, the minimum phase requirement is replaced by a robust stability property of the zero dynamics. Our control law employs a *dynamic normalization* signal to counteract the destabilizing effect of the unmodeled dynamics.

A practical feature of this redesign is that it does not require detailed information about the unmodeled dynamics subsystem, only the rate of exponential convergence and the sign of the high-frequency gain are assumed to be known. The closed-loop solutions are bounded, and converge to a compact set which can be made arbitrarily small by increasing the controller gain. If the Jacobian linearization of the zero dynamics is asymptotically stable, then the redesign recovers global asymptotic stability and local exponential stability (LES).

In Section 3.1 we introduce the class of systems to be studied, and characterize the admissible unmodeled dynamics. In Section 3.2, we present our redesign and illustrate it on an analytical example. The proofs are given in Section 3.3.

3.1 Problem Statement

We consider the system

$$\dot{X} = F(X,x) \tag{3.1}$$

$$\dot{x} = f(X,x) + g(X,x)v \tag{3.2}$$

$$\dot{\xi} = A(\xi) + Bu \tag{3.3}$$

$$v = c(\xi) + du, \tag{3.4}$$

in which $X \in \mathbb{R}^n$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}^m$, and $|g(X,x)| \geq g_0 > 0$ for all $(X,x) \in \mathbb{R}^{n+1}$. The ξ-subsystem with input $u \in \mathbb{R}$ and output $v \in \mathbb{R}$ represents unmodeled dynamics, that is, the (X,x)-subsystem with $v = u$ is the nominal system. It is assumed that all functions in (3.1)-(3.4) are C^1, and $F(\cdot,\cdot)$, $f(\cdot,\cdot)$, $A(\cdot)$ and $c(\cdot)$ vanish at zero. The stability properties analyzed are with respect to zero, which is an equilibrium for the system (3.1)-(3.4) when $u = 0$.

The main restriction on the nominal system is that the X-subsystem be globally stabilizable with x viewed as a virtual control input.

Assumption 3.1 *There exists a C^1 function $\mu(X)$ such that*

$$\dot{X} = F(X, \mu(X)) \tag{3.5}$$

is globally asymptotically stable.

We will consider nominal control laws that guarantee an ISS property for the X-subsystem. Our redesign will render such control laws robust against the destabilizing effect of the unmodeled dynamics.

For nominal control laws we employ a backstepping procedure which relies on the following fact proved by Sontag [87] (see also Isidori [27, Theorem 10.4.3]):

Proposition 3.1 *If $\mu(X)$ is as in Assumption 3.1, then there exists a C^1 function $0 < \beta(X) \leq 1$ that renders*

$$\dot{X} = F(X, \mu(X) + \beta(X)y) \tag{3.6}$$

ISS with input y. □

To proceed with the backstepping design, we note that the variable $y := (x - \mu(X))/\beta(X)$ is governed by

$$\dot{y} = \tilde{f}(X,x) + \tilde{g}(X,x)v, \tag{3.7}$$

where $|\tilde{g}(X,x)| = \frac{|g(X,x)|}{\beta(X)} \geq g_0$. Then, for the nominal system (3.1)-(3.2) with $v = u$, the control law

$$u = \alpha(X,x) = \frac{1}{\tilde{g}(X,x)}\left(-\tilde{f}(X,x) - ky\right), \quad k > 0\,, \tag{3.8}$$

results in $\dot{y} = -ky$, and guarantees GAS because of the ISS property of (3.6).

The admissible unmodeled dynamics are characterized by the following assumptions:

Assumption 3.2 *The unmodeled dynamics subsystem (3.3)-(3.4) has relative degree zero, that is* $d \neq 0$.

Assumption 3.3 *There exists a constant* $\bar{c} > 0$ *such that* $|c(\xi)| \leq \bar{c}\,|\xi|$.

Assumption 3.4 *There exists a* C^1 *Lyapunov function* $V(\xi)$ *such that*

$$v_1|\xi|^2 \leq V(\xi) \leq v_2|\xi|^2 \tag{3.9}$$

$$\frac{\partial V}{\partial \xi} A(\xi) \leq -2\delta V(\xi) \tag{3.10}$$

$$\left|\frac{\partial V}{\partial \xi}\right| \leq v_3|\xi|, \tag{3.11}$$

where δ, v_1, v_2, v_3 *are positive constants.* □

The constant $\bar{c}$ and the Lyapunov function $V(\xi)$ serve only to define the class of unmodeled dynamics. Their knowledge is not required for the redesign. By Assumption 3.4, if $u = 0$, then the unmodeled dynamics (3.3) are globally exponentially stable (GES) with the convergence rate δ. The converse of this statement is also true if $A(\xi)$ is globally Lipschitz, because, if the unmodeled dynamics subsystem is GES, then there exists[1] a $V(\xi)$ as in Assumption 3.4.

The final assumption requires a robust stability property for the zero dynamics of the subsystem (3.3)-(3.4), that is

$$\dot{z} = A(z) - \frac{1}{d} B\, c(z) =: A^0(z). \tag{3.12}$$

Assumption 3.5 *The zero dynamics subsystem (3.12) disturbed by* d_1 *and* d_2

$$\dot{z} = A^0(z + d_1) + d_2 \tag{3.13}$$

is ISS with input (d_1, d_2). □

For linear unmodeled dynamics $A(\xi) = A\xi$, $c(\xi) = C\xi$, Assumptions 3.4 and 3.5 are satisfied if the matrices A and $A^0 := A - \frac{1}{d}BC$ are both Hurwitz. This means that all relative degree zero, stable, and minimum phase linear unmodeled dynamics are admissible.

[1]See Khalil [44, Theorem 3.12], and its recent extension by Corless and Glielmo [12] which guarantees the same convergence rate for $V(\xi(t))$ as for $\xi(t)$.

3.2 Dynamic Nonlinear Damping Redesign

Using $\alpha(X,x)$ as in (3.8), the control law for system (3.1)-(3.4) is

$$\begin{aligned} u &= sgn(d)[\alpha(X,x) - \kappa\,(1 + |m| + |\alpha(X,x)|)\,\tilde{g}(X,x)y]\,, \quad \kappa > 0 &(3.14)\\ \dot{m} &= -\delta m + |u|, &(3.15)\end{aligned}$$

which only requires the knowledge of δ and the sign of d. Applying (3.14) to (3.1)-(3.4), we obtain the closed-loop system

$$\begin{aligned} \dot{X} &= F(X, \mu(X) + \beta(X)y) &(3.16)\\ \dot{y} &= \tilde{f}(X,x) &(3.17)\\ &\quad +\tilde{g}(X,x)\{c(\xi) + |d|\,[\alpha(X,x) - \kappa(1 + |m| + |\alpha(X,x)|)\,\tilde{g}(X,x)y]\}\\ \dot{\xi} &= A(\xi) + Bu\,. &(3.18)\end{aligned}$$

Adding and subtracting (3.8), we rewrite (3.17) as

$$\dot{y} = -ky + \tilde{g}(X,x)[c(\xi) + (|d| - 1)\alpha(X,x) - \kappa|d|(1 + |m| + |\alpha(X,x)|)\,\tilde{g}(X,x)y]. \qquad (3.19)$$

The three robustification terms in $-\kappa(1+|m|+|\alpha(X,x)|)\,\tilde{g}(X,x)y$ counteract the unmodeled terms $c(\xi)$ and $(|d|-1)\alpha(X,x)$ in (3.19). In particular, $-\kappa\,|\alpha(X,x)|\,\tilde{g}(X,x)y$ counteracts $(|d| - 1)\alpha(X,x)$, and can be dropped if $|d| = 1$. The remaining terms $-\kappa\,\tilde{g}(X,x)y$ and $-\kappa\,|m|\,\tilde{g}(X,x)y$ counteract $c(\xi)$, as we illustrate with the help of the following lemma:

Lemma 3.1 *Consider equations (3.15) and (3.18), and suppose $u(t) \in \mathcal{L}_\infty[0,T]$. If Assumption 3.4 holds, then there exist constants θ_1, $\theta_2 > 0$ such that, for all $t \in [0,T]$,*

$$|\xi(t)| \le \theta_1\,(|\xi(0)| + |m(0)|)\,e^{-\delta t} + \theta_2|m(t)|. \qquad (3.20)$$

□

The proof is given in Section 3.3. For every time interval $[0,T]$ in which the closed-loop solutions exist, Lemma 3.1 implies that

$$|c(\xi(t))| \le \bar{c}|\xi(t)| \le \bar{c}\theta_1(|\xi(0)| + |m(0)|)e^{-\delta t} + \bar{c}\theta_2|m(t)|. \qquad (3.21)$$

Thus, $-\kappa\,\tilde{g}(X,x)y$ and $-\kappa\,|m|\,\tilde{g}(X,x)y$ counteract $\bar{c}\theta_1(|\xi(0)|+|m(0)|)e^{-\delta t}$ and $\bar{c}\theta_2|m(t)|$, respectively.

Our redesign guarantees boundedness of the states (X,x,ξ) and the signal m for any $\kappa > 0$. Moreover, the states (X,x,ξ) converge to a compact set around the origin, which can be made arbitrarily small by increasing κ.

Theorem 3.1 *Consider the system (3.1)-(3.4), and suppose Assumptions 3.2-3.5 hold. If $\alpha(X,x)$ is as in (3.8), then the control law (3.14)-(3.15) guarantees that the closed-loop solutions $(X(t),x(t),\xi(t),m(t))$ are bounded for all $t \geq 0$. Moreover, there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ with the property that, for any given $\epsilon > 0$, we can find $\kappa^\star$ such that for all $\kappa \geq \kappa^\star$,*

$$|(X(t),x(t),\xi(t))| \leq \max\{\beta(|(X(0),x(0),\xi(0),m(0))|,t)\,,\ \epsilon\} \qquad \forall t \geq 0\,. \tag{3.22}$$

If, in addition, the Jacobian linearizations of $F(X,\mu(X))$ in (3.5) and $A^0(z)$ in (3.12) are asymptotically stable at zero, then the closed loop system (3.1)-(3.4),(3.14)-(3.15) is globally asymptotically stable and locally exponentially stable for sufficiently large κ. □

The proof is given in Section 3.3. Inequality (3.22) indicates that the state variables of the plant (3.1)-(3.4) converge to a ball of radius ϵ, where ϵ can be rendered arbitrarily small by increasing κ. It is important to note that increasing κ does not cause *peaking* in the transients because $\beta(\cdot,\cdot)$ is independent of κ.

Example 3.1 *The redesign for the system*

$$\begin{aligned} \dot{X} &= X^3 + x \\ \dot{x} &= x^2 + (1+X^2)v \qquad (3.23)\\ \dot{\xi}_1 &= -2\xi_1 - \xi_1^3 + \xi_2 \\ \dot{\xi}_2 &= -2\xi_2 + u \\ v &= \xi_1 + \frac{-\xi_2 + 10\xi_2^3}{1+\xi_2^2} + u \qquad (3.24) \end{aligned}$$

starts with the design of the nominal control law $\alpha(X,x)$. The X-subsystem satisfies Assumption 3.1 with $\mu(X) = -X - X^3$, and Proposition 3.1 holds with $\beta(X) = 1$. The y-subsystem is

$$\dot{y} = \tilde{f}(X,x) + \tilde{g}(X,x)v = x^2 + (1+3X^2)(X^3+x) + (1+X^2)v \tag{3.25}$$

and, hence,

$$\alpha(X,x) = \frac{1}{(1+X^2)}\left(-x^2 - (1+3X^2)(X^3+x) - ky\right), \quad k > 0. \tag{3.26}$$

The remaining task is to show that the unmodeled dynamics satisfy Assumptions 3.2-3.5. Assumptions 3.2 and 3.3 hold because, from (3.24),

$$d = 1 \quad \textit{and} \quad c(\xi) = \xi_1 + \frac{-\xi_2 + 10\xi_2^3}{1+\xi_2^2}, \tag{3.27}$$

and Assumption 3.4 is verified with $V(\xi) = \frac{1}{2}(\xi_1^2 + \xi_2^2)$ *and* $\delta = 1.5$. *To verify Assumption 3.5 for the zero dynamics*

$$A^0(z) = \begin{bmatrix} -2z_1 - z_1^3 + z_2 \\ -z_1 - 2z_2 + \frac{z_2 - 10z_2^3}{1+z_2^2} \end{bmatrix}, \tag{3.28}$$

we use the ISS Lyapunov function $V_1 = \frac{1}{2}z^T z$ *and note from (3.13) that*

$$\dot{V}_1 = z^T[A^0(z + d_1) - A^0(d_1)] + z^T[A^0(d_1) + d_2], \tag{3.29}$$

which, in view of

$$A^0(z + d_1) - A^0(d_1) = \int_0^1 \left[\frac{\partial A^0}{\partial s}\right]_{s=d_1+\lambda z} z \, d\lambda \tag{3.30}$$

yields

$$\dot{V}_1 = \frac{1}{2}\left\{\int_0^1 z^T \left(\left[\frac{\partial A^0}{\partial s}\right] + \left[\frac{\partial A^0}{\partial s}\right]^T\right)_{s=d_1+\lambda z} z \, d\lambda\right\} + z^T[A^0(d_1) + d_2]. \tag{3.31}$$

It follows from (3.28) that

$$\left[\frac{\partial A^0}{\partial z}\right] + \left[\frac{\partial A^0}{\partial z}\right]^T \leq -2I \quad \forall z \in I\!R^2 \tag{3.32}$$

and, hence,

$$\dot{V}_1 \leq -z^T z + z^T[A^0(d_1) + d_2] \leq -V_1 + \frac{1}{2}|[A^0(d_1) + d_2]|^2, \tag{3.33}$$

which is an ISS property with input (d_1, d_2).

Having verified that Assumptions 3.2-3.5 hold, we substitute $\tilde{g}(X, x) = 1 + X^2$, $y = x + X + X^3$, $sgn(d) = 1$ *and* $\delta = 1.5$ *in (3.14)-(3.15), and obtain the control law*

$$u = \alpha(X, x) - \kappa\left(1 + |m| + |\alpha(X, x)|\right)(1 + X^2)(x + X + X^3), \ \kappa > 0 \tag{3.34}$$

$$\dot{m} = -1.5m + |u|, \tag{3.35}$$

which completes the redesign of the nominal control law $\alpha(X, x)$ *in (3.26). The closed-loop system (3.23)-(3.24) is GAS and LES for sufficiently large* κ, *because the Jacobian linearizations of* $A^0(z)$ *and* $F(X, \mu(X)) = -X$ *are asymptotically stable.* □

3.3 Proofs

3.3.1 Proof of Lemma 3.1

Starting with $V(\xi)$ as in Assumption 3.4, we use the Lyapunov function $W(\xi) := \sqrt{V(\xi)}$ which is not differentiable at the origin, but is locally Lipschitz because from (3.9) and (3.11),

$$\left|\frac{\partial W}{\partial \xi}\right| = \left|\frac{1}{2\sqrt{V}}\frac{\partial V}{\partial \xi}\right| \leq \frac{v_3}{2\sqrt{v_1}} \quad \forall \xi \neq 0. \tag{3.36}$$

We need the following result of Teel and Praly [97], which makes use of the generalized directional derivative of Clarke [11].

Proposition 3.2 *Suppose $W(\xi)$ is locally Lipschitz, $f(\xi,u)$ and $\tilde{\alpha}(\xi,u)$ are continuous, and*

$$\frac{\partial W}{\partial \xi} f(\xi,u) \leq \tilde{\alpha}(\xi,u) \quad \forall u, \forall \xi \notin \Omega, \tag{3.37}$$

where Ω is the set in which W is not differentiable. Let $u(t)$ be a function defined on $[0,T]$, and let $\xi(t)$ be an absolutely continuous function satisfying $\dot{\xi} = f(\xi(t),u(t))$ on $[0,T]$. Then, for almost all $t \in [0,T]$,

$$\dot{W} \leq \tilde{\alpha}(\xi(t),u(t)). \tag{3.38}$$

□

To evaluate (3.37), we note from (3.10) and (3.36) that, for all $\xi \neq 0$,

$$\frac{\partial W}{\partial \xi}(A(\xi)+Bu) = \frac{1}{2\sqrt{V}}\frac{\partial V}{\partial \xi}(A(\xi)+Bu) \leq -\delta W + \frac{v_3}{2\sqrt{v_1}}\|B\|\,|u|. \tag{3.39}$$

Denoting $v_4 := \frac{v_3}{2\sqrt{v_1}}\|B\|$, we conclude from Proposition 3.2 that, for almost all $t \in [0,T]$,

$$\dot{W} \leq -\delta W + v_4|u| \tag{3.40}$$

and, hence,

$$W(\xi(t)) \leq W(\xi(0))e^{-\delta t} + v_4\int_0^t e^{-\delta(t-\tau)}|u(\tau)|d\tau \quad \forall t \in [0,T]. \tag{3.41}$$

Substituting

$$\int_0^t e^{-\delta(t-\tau)}|u(\tau)|d\tau = m(t) - m(0)e^{-\delta t}, \tag{3.42}$$

obtained from the solution of (3.15), we get

$$W(\xi) \leq [W(\xi(0)) - v_4 m(0)]e^{-\delta t} + v_4 m(t) \quad \forall t \in [0, T]. \tag{3.43}$$

Using (3.9), we obtain

$$\sqrt{v_1}|\xi(t)| \leq (\sqrt{v_2}|\xi(0)| + v_4|m(0)|)e^{-\delta t} + v_4|m(t)|, \tag{3.44}$$

from which (3.20) follows with $\theta_1 = \frac{1}{\sqrt{v_1}} \max\{\sqrt{v_2}, v_4\}$ and $\theta_2 = \frac{v_4}{\sqrt{v_1}}$. □

3.3.2 Proof of Theorem 3.1

We divide the proof into four parts. First, we prove that in (3.1)-(3.4) there are no finite escape times, and that $X(t)$ and $x(t)$ are bounded. Next, we prove that $\xi(t)$ is also bounded, and derive (3.22). In the third part we prove the boundedness of $m(t)$. Finally, we prove GAS and LES under the additional assumption that the Jacobian linearizations of $A^0(z)$ and $F(X, \mu(X))$ are asymptotically stable.

Part 1: $(X, x) \in \mathcal{L}_\infty$, $(\xi, m) \in \mathcal{L}_\infty^e$

The closed-loop system is locally Lipschitz and, hence, solutions exist and are unique. To prove the absence of finite escape times, we analyze the solutions on the compact interval $[0, T]$ where T is in the maximal interval of existence, and show that they are bounded by a continuous function of T on $[0, \infty)$. From (3.19),

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}y^2\right) \quad \leq \quad & -ky^2 + |\tilde{g}(X,x)y|\,|c(\xi)| + |\tilde{g}(X,x)y|\,|(|d|-1)|\,|\alpha(X,x)| \\ & -\kappa|d|(1+|m|+|\alpha(X,x)|)\,|\tilde{g}(X,x)y|^2. \end{aligned} \tag{3.45}$$

Substituting (3.21) and $|\tilde{g}(X,x)y| \geq g_0|y|$, and rearranging terms, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}y^2\right) \quad \leq \quad & -ky^2 - \kappa|d|\,|m|\,|\tilde{g}(X,x)y|\,\left(g_0|y| - \frac{\bar{c}\theta_2}{\kappa|d|}\right) \\ & -\kappa|d|\,|\tilde{g}(X,x)y|\,\left(g_0|y| - \frac{\bar{c}\theta_1}{\kappa|d|}\left(|\xi(0)| + |m(0)|\right)e^{-\delta t}\right) \\ & -\kappa|d|\,|\alpha(X,x)|\,|\tilde{g}(X,x)y|\,\left(g_0|y| - \frac{|(|d|-1)|}{\kappa|d|}\right). \end{aligned} \tag{3.46}$$

Defining

$$r := \frac{1}{g_0}\max\left\{\frac{\bar{c}\theta_2}{|d|},\ \frac{|(|d|-1)|}{|d|}\right\}, \quad \zeta(t) := \max\left\{\frac{\bar{c}\theta_1}{\kappa|d|g_0}\left(|\xi(0)| + |m(0)|\right)e^{-\delta t},\ \frac{r}{\kappa}\right\}, \tag{3.47}$$

we note from (3.46) that

$$|y| \geq |\zeta(t)| \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}y^2\right) \leq -ky^2. \tag{3.48}$$

This means that

$$|y(t)| \leq \max\left\{|y(0)|e^{-kt}, \sup_{0\leq\tau\leq t}|\zeta(\tau)|\right\} \quad \forall t \in [0,T], \tag{3.49}$$

which is an ISS property with input $\zeta(t)$. It follows from (3.47) that

$$|\zeta(t)| \leq \max\left\{|\zeta(0)|e^{-\delta t}, \frac{r}{\kappa}\right\}, \tag{3.50}$$

that is $\zeta(t)$ is ISS with input $\frac{r}{\kappa}$. Moreover, the X-subsystem (3.6) is ISS with input y. Recalling that the cascade interconnection of ISS systems is ISS (see Sontag [86]), we conclude that there exists a class-$\mathcal{K}$ function $\gamma_0(\cdot)$ and a class-$\mathcal{KL}$ function $\beta_0(\cdot,\cdot)$ such that

$$|(X(t),x(t))| \leq \max\left\{\beta_0(|(X(0),x(0),\zeta(0))|,t),\ \gamma_0\left(\frac{1}{\kappa}\right)\right\} \quad \forall t \in [0,T]. \tag{3.51}$$

This gives an upper bound on $(X(t),x(t))$ which is independent of T. Using this upper bound in (3.14)-(3.15), we can find a constant N such that for almost all $t \in [0,T]$,

$$\frac{d}{dt}|m| \leq N|m| + N, \tag{3.52}$$

which implies that $|m(t)|$ is bounded by a continuous function of T on $[0,\infty)$. Likewise, a continuous bound can be derived for $|\xi(t)|$ using (3.21). Since the closed-loop signals (X,x,ξ,m) are bounded by a continuous function of T on $[0,\infty)$, the maximal interval of existence is infinite, that is $(X,x,\xi,m) \in \mathcal{L}^e_\infty$. Moreover (3.51) holds for all $t \geq 0$, which proves that $(X,x) \in \mathcal{L}_\infty$.

Part 2: $\xi \in \mathcal{L}_\infty$

Our derivations so far have not relied on Assumption 3.5. We now employ Assumption 3.5 to prove that $\xi \in \mathcal{L}_\infty$. To eliminate u from (3.3), we use (3.4) and obtain

$$\dot{\xi} = A^0(\xi) + B^0 v, \tag{3.53}$$

where $A^0(\xi) := A(\xi) - \frac{1}{d}Bc(\xi)$ and $B^0 := \frac{1}{d}B$. Then, it follows from (3.2) and (3.53) that the variable

$$z := \xi - B^0 \int_0^x \frac{d\sigma}{g(X,\sigma)} \tag{3.54}$$

is governed by

$$\dot{z} = A^0\left(z + B^0\int_0^x \frac{d\sigma}{g(X,\sigma)}\right) - B^0\frac{f(X,x)}{g(X,x)}, \tag{3.55}$$

which is (3.13) with $d_1 = B^0\int_0^x \frac{d\sigma}{g(X,\sigma)}$, and $d_2 = -B^0\frac{f(X,x)}{g(X,x)}$. Since d_1 and d_2 are continuous functions of (X,x), and vanish at zero, it follows from Assumption 3.5 and the ISS property of the cascade of ISS systems that there exist a class-$\mathcal{KL}$ function $\beta_1(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma_1(\cdot)$ such that, for all $t \geq 0$,

$$|(X(t),x(t),\xi(t))| \leq \max\left\{\beta_1(|(X(0),x(0),\xi(0),\zeta(0))|,t),\ \gamma_1\left(\frac{1}{\kappa}\right)\right\}. \tag{3.56}$$

This proves that $\xi \in \mathcal{L}_\infty$. To prove that (3.22) holds, we need to eliminate $\zeta(0)$ from (3.56). To this end, we let $\bar{\kappa} > 0$ and observe from (3.47) that for all $\kappa \geq \bar{\kappa}$,

$$|\zeta(0)| \leq \max\left\{\frac{\bar{c}\theta_1}{\bar{\kappa}|d|g_0}(|\xi(0)| + |m(0)|), \frac{r}{\kappa}\right\}. \tag{3.57}$$

Using this inequality in (3.56), we can find a class-$\mathcal{KL}$ function $\beta_{\bar{\kappa}}(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that for all $\kappa \geq \bar{\kappa}$,

$$|(X(t),x(t),\xi(t))| \leq \max\left\{\beta_{\bar{\kappa}}(|(X(0),x(0),\xi(0),m(0))|,t),\ \gamma\left(\frac{1}{\kappa}\right)\right\} \quad \forall t \geq 0. \tag{3.58}$$

Then, (3.22) follows by setting $\beta(\cdot,\cdot) = \beta_{\bar{\kappa}}(\cdot,\cdot)$ and $\kappa^\star = \max\{\frac{1}{\gamma^{-1}(\epsilon)}, \bar{\kappa}\}$.

Part 3: $m \in \mathcal{L}_\infty$

To prove boundedness of $m(t)$, we consider the differential equations (3.15) and (3.19), and analyze their solutions in the half-plane $H^+ := \{(y,m)|\, m \geq 0\}$ which is forward invariant from (3.15). It is sufficient to analyze solutions in H^+ because (3.15) implies that solutions satisfy $|m(t)| \leq |m(0)|e^{-\delta t}$ as long as they remain outside H^+. We use the function

$$U(y,m) = \frac{1}{2}y^2 + \theta m, \tag{3.59}$$

where the positive constant θ is to be determined. We first note that there exist class-$\mathcal{K}_\infty$ functions $u_1(\cdot)$ and $u_2(\cdot)$ such that, $\forall\,(y,m) \in H^+$,

$$u_1(|(y,m)|) \leq U(y,m) \leq u_2(|(y,m)|). \tag{3.60}$$

Next, from (3.15) and (3.19), $\dot{U}(y,m)$ in H^+ satisfies

$$\dot{U} \le -ky^2 - \kappa|d|\,|\tilde{g}(X,x)y|^2 m + \theta(-\delta m + |u|) + |\tilde{g}(X,x)y|(|c(\xi)| + |(|d|-1)|\,|\alpha(X,x)|). \tag{3.61}$$

Substituting

$$|u| \le |\alpha(X,x)| + \kappa(1+|\alpha(X,x)|)|\tilde{g}(X,x)y| + \kappa|\tilde{g}(X,x)y|\,m \tag{3.62}$$

in (3.61), and defining

$$R := |\tilde{g}(X,x)y|(|c(\xi)| + |(|d|-1)|\,|\alpha(X,x)|) + \theta(|\alpha(X,x)| + \kappa(1+|\alpha(X,x)|)|\tilde{g}(X,x)y|), \tag{3.63}$$

we obtain

$$\dot{U} \le -ky^2 - \kappa|d|\left(|\tilde{g}(X,x)y|^2 - \theta\frac{1}{|d|}|\tilde{g}(X,x)y| + \theta\frac{\delta}{\kappa|d|}\right)m + R. \tag{3.64}$$

We pick $\theta > 0$ small enough to guarantee $p^2 - \theta\frac{1}{|d|}p + \theta\frac{\delta}{\kappa|d|} > 0$, $\forall p \in I\!R$, so that we can find $q_1 > 0$ such that

$$\left(|\tilde{g}(X,x)y|^2 - \theta\frac{1}{|d|}|\tilde{g}(X,x)y| + \theta\frac{\delta}{\kappa|d|}\right) \ge q_1. \tag{3.65}$$

Substituting in (3.64), we get

$$\dot{U} \le -q_2 U + R, \tag{3.66}$$

where $q_2 = \min\{2k, \frac{\kappa|d|q_1}{\theta}\}$. Since $R(t)$ is bounded, (3.66) implies that $U(t)$ is bounded. Then, $m \in \mathcal{L}_\infty$ follows from (3.60).

Part 4: GAS and LES

Defining $A_L^0 = \frac{\partial A^0(z)}{\partial z}|_{z=0}$ and $F_L = \frac{\partial F(X,\mu(X))}{\partial X}|_{X=0}$, the Jacobian linearization of the (X,y,z)-subsystem (3.16), (3.17) and (3.55) is

$$\begin{aligned}
\dot{z} &= A_L^0 z + B_1 X + b_2 y \\
\dot{X} &= F_L X + b_3 y \\
\dot{y} &= -\kappa\, c_1(1+|m|)y + c_2 y + b_4 z + b_5 X.
\end{aligned} \tag{3.67}$$

Since A_L^0 and F_L are Hurwitz, we let $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$ satisfy ${A_L^0}^T P_1 + P_1 A_L^0 = -I$, $F_L^T P_2 + P_2 F_L = -I$, and use

$$S = z^T P_1 z + \gamma X^T P_2 X + y^2, \quad \gamma > 0, \tag{3.68}$$

as a Lyapunov function for the nonlinear (X, y, z)-subsystem (3.16),(3.17),(3.55). The expression for $\dot{S}$ shows that by selecting κ and γ sufficiently large, we can find constants $\varrho > 0$ and $\vartheta > 0$ such that $\dot{S} \leq -\varrho S$ is satisfied for all (X, y, z) in the set $\Omega_\vartheta := \{(X, y, z) : S \leq \vartheta\}$. Thus, if a trajectory $(X(t), y(t), z(t), m(t))$ enters $\Omega_\vartheta \times I\!R$, then $(X(t), y(t), z(t)) \to 0$ exponentially as $t \to \infty$. This also ensures $m(t) \to 0$ because of (3.66), where R defined by (3.63) as a function of (X, y, z) vanishes at zero. Choosing $\epsilon > 0$ sufficiently small in (3.22), we guarantee that the solutions (X, y, z, m) enter $\Omega_\vartheta \times I\!R$ in finite time and, hence, $(X(t), y(t), z(t), m(t)) \to 0$. Finally, LES is proved with the help of the Lyapunov function $\bar{\gamma}S + U$, in which U is as in (3.66) and the constant $\bar{\gamma} > 0$ is sufficiently large. □

3.4 Summary

The redesign in this chapter renders a class of nonlinear control laws robust against input unmodeled dynamics which are relative degree zero, globally exponentially stable, and have ISS zero dynamics. For linear unmodeled dynamics, the latter two conditions are equivalent to stability and minimum phase properties, which are less restrictive than the conditions required by previous redesigns. Nonlinear unmodeled dynamics considered here also differ from those in other redesigns.

Chapter 4

Nested Saturation Redesign

The redesigns discussed so far require that the unmodeled dynamics be relative-degree zero and minimum-phase. This restricts the applicability of these redesigns because most physical examples of unmodeled dynamics, like actuator models, have a higher relative degree.

For systems in *feedforward* form, the *nested saturation* design of Teel [92, 93] employs saturation elements to guarantee a small-gain property. In the absence of unmodeled dynamics, Teel's design achieves GAS and LES.

In this chapter we redesign nested saturation control laws to guarantee GAS and LES in the presence of input unmodeled dynamics. The redesign is applicable to a large class of unmodeled dynamics, not restricted to be relative degree zero or minimum phase. Using the *asymptotic small-gain theorem* of Teel [93], we prove not only GAS, but also an asymptotic gain property from small disturbances to the states.

4.1 Problem Statement

We consider the locally Lipschitz system

$$\begin{aligned} \dot{x}_n &= x_{n-1} + g_n(x_1, x_2, \cdots, x_{n-1}, v, d) \\ \vdots &= \vdots \\ \dot{x}_2 &= x_1 + g_2(x_1, v, d) \\ \dot{x}_1 &= v + g_1(v, d) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \dot{\xi} &= q(\xi, u, d) \\ v &= p(\xi, u, d), \end{aligned} \tag{4.2}$$

where $d \in I\!R^m$ represents disturbances, $\xi \in I\!R^p$, $u, v \in I\!R$, the functions $q(\xi, u, d)$ and $p(\xi, u, d)$ vanish at zero, and g_i's satisfy

$$\begin{aligned} g_1(v, 0) &= o(v) \\ g_i(x_1, \cdots, x_{i-1}, v, 0) &= o(x_1, \cdots, x_{i-1}, v), \quad i = 2, \cdots, n, \end{aligned} \tag{4.3}$$

where the notation $g(v) = o(v)$ means that

$$\lim_{|v| \to 0} \frac{|g(v)|}{|v|} = 0\,. \tag{4.4}$$

For the nominal system, that is (4.1) with $v = u$ and $d \equiv 0$, the nested saturation design guarantees GAS and LES. However, the following example illustrates the loss of stability due to unmodeled dynamics:

Example 4.1 *For the system*

$$\begin{aligned} \dot{x}_2 &= x_1 + g_2(x_1, v) \\ \dot{x}_1 &= v + g_1(v), \end{aligned} \tag{4.5}$$

the nested saturation control law is

$$v = u = -\phi_1(y_1 + \phi_2(y_2)), \tag{4.6}$$

where $y_1 = x_1$, $y_2 = x_1 + x_2$, *and* $\phi_i(\cdot)$*'s are saturation functions with saturation levels selected according to Teel [92].*

In the presence of the input unmodeled dynamics

$$v(s) = \frac{-s+1}{s^2+s+1}\, u(s), \tag{4.7}$$

the resulting closed-loop system is unstable as verified from the Jacobian linearization. □

In this example instability is established from the linearization, therefore it cannot be prevented by adjusting the saturation levels. To guarantee robustness against unmodeled dynamics, we introduce the controller gains $k_1, \cdots, k_n$ in the redesigned control law

$$u = -\phi_1(k_1 x_1 + \phi_2(k_2 x_2 + \cdots + \phi_n(k_n x_n) \cdots)), \tag{4.8}$$

where $\phi_i(\cdot)$'s are saturation functions as defined in (1.56). We provide a procedure for selection of the gains k_i and the saturation levels λ_i to ensure robustness.

For unmodeled dynamics satisfying Properties 4.1 and 4.2 below, our redesign achieves an asymptotic gain from the disturbance d to the state (x, ξ) when

$$\|d\|_a := \limsup_{t\to\infty} |d(t)|$$

is sufficiently small. In particular, if $\|d\|_a = 0$, then $x(t)$ and $\xi(t)$ converge to the origin. For $d(t) \equiv 0$, the origin $(x, \xi) = (0, 0)$ is GAS and LES.

Property 4.1 *The vector field $q(\xi, u, 0)$ and the output function $p(\xi, u, 0)$ are differentiable at the origin and the Jacobian linearization*

$$\begin{aligned} A &:= \left[\frac{\partial q(\xi,u,0)}{\partial \xi}\right]_{(0,0)} & B &:= \left[\frac{\partial q(\xi,u,0)}{\partial u}\right]_{(0,0)} \\ C &:= \left[\frac{\partial p(\xi,u,0)}{\partial \xi}\right]_{(0,0)} & D &:= \left[\frac{\partial p(\xi,u,0)}{\partial u}\right]_{(0,0)} \end{aligned} \tag{4.9}$$

is such that

1. *The system matrix A is Hurwitz,*
2. *The dc gain $\delta = D - CA^{-1}B > 0$ is positive.*

Property 4.2 *For $\dot{\xi} = q(\xi, u, d)$, there exist constants $c_u, c_d, \Delta_d \geq 0$, $\Delta_u > 0$ such that, if $u(t)$, $d(t)$ are locally bounded, $\|u\|_a \leq \Delta_u$, $\|d\|_a \leq \Delta_d$, then, for each initial condition $\xi(0)$, the solution $\xi(t)$ exists for all $t \geq 0$, and*

$$\|\xi\|_a \leq \max\{c_u\|u\|_a, c_d\|d\|_a\}. \tag{4.10}$$

Inequality (4.10) expresses an asymptotic gain from the input (u, d) to the state ξ. If the ξ-subsystem has a linear *input-to-state stability gain* as in Sontag [86], then Property 4.2 is satisfied with $\Delta_u = \Delta_d = \infty$.

4.2 Nested Saturation Redesign

Substituting $v = p(\xi, u, d)$ and using (4.9), we rewrite the system (4.1),(4.2) as

$$\begin{aligned} \dot{x}_n &= x_{n-1} + G_n(x_1, x_2, \cdots, x_{n-1}, \xi, u, d) \\ \vdots &= \vdots \\ \dot{x}_2 &= x_1 + G_2(x_1, \xi, u, d) \\ \dot{x}_1 &= C\xi + Du + G_1(\xi, u, d) \\ \dot{\xi} &= q(\xi, u, d), \end{aligned} \tag{4.11}$$

where $G_1(\xi,u,d) := p(\xi,u,d) - C\xi - Du + g_1(p(\xi,u,d),d)$ and, for $i = 2,\cdots,n$,

$$G_i(x_1,\cdots,x_{i-1},\xi,u,d) := g_i(x_1,\cdots,x_{i-1},p(\xi,u,d),d). \tag{4.12}$$

Because of (4.3), G_i's satisfy

$$\begin{aligned} G_1(\xi,u,0) &= o(\xi,u) \\ G_i(x_1,\cdots,x_{i-1},\xi,u,0) &= o(x_1,\cdots,x_{i-1},\xi,u), \quad i = 2,\cdots,n. \end{aligned} \tag{4.13}$$

The saturation levels $\lambda_i > 0$ and the gains $k_i > 0$ in (4.8) are designed recursively: In the first step we set

$$u = -\phi_1(k_1 x_1 - u_1), \tag{4.14}$$

where u_1 is to be designed in the next step. Selecting $\lambda_1 > 0$ and $k_1 > 0$ according to Lemma 4.1 below, we guarantee that the composite system

$$\begin{aligned} \dot{\xi}_1 &:= \begin{bmatrix} \dot{x}_1 \\ \dot{\xi} \end{bmatrix} = q_1(\xi_1,u_1,d) := \left.\begin{bmatrix} C\xi + Du + G_1(\xi,u,d) \\ q(\xi,u,d) \end{bmatrix}\right|_{u=-\phi_1(k_1x_1-u_1)} \\ v_1 &:= x_1 = [1\ 0\cdots\ 0]\xi_1 =: C_1\xi_1 \end{aligned} \tag{4.15}$$

with input (u_1,d) and output v_1 possesses the same properties as the unmodeled dynamics subsystem, that is, Property 4.2 is satisfied with q,ξ,u replaced by q_1,ξ_1,u_1, and Property 4.1 is satisfied with the Jacobian linearization A_1,B_1 of the vector field $q_1(\xi_1,u_1,0)$, $D_1 = 0$ and C_1 as in (4.15).

In the second step we treat the $\xi_1 = [x_1,\xi^T]^T$ subsystem of (4.11) with input (u_1,d) and output $v_1 = C_1\xi_1$ as the "virtual" unmodeled dynamics subsystem for the $(x_2,\cdots,x_n)$ subsystem and rewrite (4.11) as

$$\begin{aligned} \dot{x}_n &= x_{n-1} + G_{n,2}(x_2,\cdots,x_{n-1},\xi_1,u_1,d) \\ \vdots &= \vdots \\ \dot{x}_2 &= C_1\xi_1 + G_{2,2}(\xi_1,u_1,d) \\ \dot{\xi}_1 &= q_1(\xi_1,u_1,d), \end{aligned} \tag{4.16}$$

where, for $j = 2,\cdots,n$, the function $G_{j,2}$ is obtained by substituting (4.14) in G_j. We note that

$$\begin{aligned} G_{2,2}(\xi_1,u_1,0) &= o(\xi_1,u_1) \\ G_{j,2}(x_2,\cdots,x_{j-1},\xi_1,u_1,0) &= o(x_2,\cdots,x_{j-1},\xi_1,u_1), \quad j = 3,\cdots,n, \end{aligned} \tag{4.17}$$

and proceed to design

$$u_1 = -\phi_2(k_2 x_2 - u_2) \tag{4.18}$$

such that the composite system

$$\begin{aligned}\dot{\xi}_2 := \begin{bmatrix} \dot{x}_2 \\ \dot{\xi}_1 \end{bmatrix} = q_2(\xi_2, u_2, d) := \left.\begin{bmatrix} C_1\xi_1 + G_{2,2}(\xi_1, u_1, d) \\ q_1(\xi_1, u_1, d) \end{bmatrix}\right|_{u_1=-\phi_2(k_2x_2-u_2)} \\ v_2 := x_2 = [\,1\ 0\ 0\ \cdots\ 0\,]\xi_2 =: C_2\xi_2 \end{aligned} \tag{4.19}$$

with input (u_2, d) and output v_2 satisfies Properties 1 and 2.

This recursive design relies on the assumption that at the ith step λ_i and k_i can be selected such that the ξ_i-subsystem with input (u_i, d) and output v_i satisfies Properties 1 and 2. We now prove that this is guaranteed by selecting both $\lambda_i > 0$ and $k_i > 0$ sufficiently small.

Lemma 4.1 *Consider the system (4.11) and let ξ_i be defined by $\xi_0 := \xi$, $\xi_i := [x_i, \xi_{i-1}^T]^T$, $i = 1, \cdots, n$. Let the control u be constructed recursively by*

$$u_{i-1} = -\phi_i(k_i x_i - u_i), \quad i = 1, \cdots, n, \tag{4.20}$$

where $\phi_i(\cdot)$ is a saturation function with saturation level λ_i, $u_0 = u$ and $u_n := 0$. Let $q_0 := q$, $G_{1,1} := G_1$, $C_0 := C$, $D_0 := D$, and $C_i := [\,1\ 0_{1\times(p+i-1)}\,]$, $D_i := 0$, $i = 1, \cdots, n$, and write the ξ_i-subsystem of (4.11) as

$$\begin{aligned} \dot{x}_i &= v_{i-1} + G_{i,i}(\xi_{i-1}, u_{i-1}, d) \\ \dot{\xi}_{i-1} &= q_{i-1}(\xi_{i-1}, u_{i-1}, d), \end{aligned} \tag{4.21}$$

where $G_{i,i}(\xi_{i-1}, u_{i-1}, 0) = o(\xi_{i-1}, u_{i-1})$.

If the ξ_{i-1}-subsystem with input (u_{i-1}, d) and output

$$v_{i-1} = C_{i-1}\xi_{i-1} + D_{i-1}u_{i-1} \tag{4.22}$$

possesses Properties 1 and 2, then there exist $\lambda_i^\star > 0$ and $k_i^\star > 0$ such that, with $\lambda_i \in (0, \lambda_i^\star]$ and $k_i \in (0, k_i^\star]$ in (4.20), the ξ_i-subsystem

$$\begin{aligned} \dot{\xi}_i := \begin{bmatrix} \dot{x}_i \\ \dot{\xi}_{i-1} \end{bmatrix} &= q_i(\xi_i, u_i, d) \\ &:= \left.\begin{bmatrix} C_{i-1}\xi_{i-1} + D_{i-1}u_{i-1} + G_{i,i}(\xi_{i-1}, u_{i-1}, d) \\ q_{i-1}(\xi_{i-1}, u_{i-1}, d) \end{bmatrix}\right|_{u_{i-1}=-\phi_i(k_ix_i-u_i)} \end{aligned} \tag{4.23}$$

with input (u_i, d) *and output*

$$v_i := x_i = C_i \xi_i + D_i u_i \tag{4.24}$$

satisfies Properties 1 and 2. □

The proof is given in Section 4.3. Our control law is constructed by recursively applying Lemma 4.1 for $i = 1, \cdots, n$ to select λ_i and k_i. At the nth step, the saturation level λ_n and the gain k_n in

$$u_{n-1} := -\phi_n(k_n x_n) \tag{4.25}$$

guarantee that the closed-loop system

$$\dot{\xi}_n := \begin{bmatrix} \dot{x}_n \\ \dot{\xi}_{n-1} \end{bmatrix} = q_n(\xi_n, d) := \left. \begin{bmatrix} x_{n-1} + G_{n,n}(\xi_{n-1}, u_{n-1}, d) \\ q_{n-1}(\xi_{n-1}, u_{n-1}, d) \end{bmatrix} \right|_{u_{n-1}=-\phi_n(k_n x_n)} \tag{4.26}$$

possesses Properties 1 and 2, which yield the following result:

Theorem 4.1 *Consider the system (4.1) with the unmodeled dynamics (4.2) satisfying Properties 1 and 2. If the control law (4.8) is constructed by recursively applying Lemma 4.1 for* $i = 1, \cdots, n$*, then there exist* $\Delta \geq 0$ *and* $c \geq 0$ *such that,*

1. If $d(t)$ *is locally bounded and* $\|d\|_a \leq \Delta$*, then* $(x(t), \xi(t))$ *exist for all* $t \geq 0$ *and*

$$\|(x, \xi)\|_a \leq c\|d\|_a. \tag{4.27}$$

2. If $d(t) \equiv 0$*, then the origin is globally asymptotically stable and locally exponentially stable.*

Proof: Inequality (4.27) follows directly from Property 4.2. If $\|d\|_a = 0$, then (4.27) guarantees global attractivity of the origin $(x, \xi) = (0, 0)$. If, in addition, $d(t) \equiv 0$, we have LES because the Jacobian linearization A_n of $q_n(\xi_n, 0)$ is Hurwitz by Property 4.1. Since the resulting closed-loop system is time-invariant, global attractivity and LES together imply GAS. □

It is important to note that this redesign does not require detailed knowledge of the unmodeled dynamics (4.2). Indeed, from the proof of Lemma 4.1, the selection of λ_i and k_i is based on upper bounds on $\delta = D - CA^{-1}B$, $|CA^{-1}|$, c_u and Δ_u in Properties 1 and 2, and, on functions that form upper bounds on $|p(\xi, u, 0)|$, $|\tilde{q}(\xi, u, 0)|$

and $|\tilde{p}(\xi, u, 0)|$, where $\tilde{q}$ and $\tilde{p}$ represent the deviations of $q(\xi, u, 0)$ and $p(\xi, u, 0)$ from their linearizations $A\xi + Bu$ and $C\xi + Du$, respectively. Therefore, the control law constructed as in Theorem 4.1 guarantees robustness against a class of unmodeled dynamics characterized by these upper bounds.

Example 4.2 *We now apply the redesign to the system of Example 4.1, with $g_2(x_1, v) = x_1^2 + v^2$ and $g_1(v) = v^2$. The linear unmodeled dynamics subsystem (4.7) is Hurwitz and has a positive dc gain $\delta = 1$, thus it satisfies Properties 1 and 2. The redesigned nested saturation control law*

$$u = -\phi_1(k_1 x_1 + \phi_2(k_2 x_2)), \tag{4.28}$$

with parameters $\lambda_1 = 0.5$, $k_1 = 0.4$, $\lambda_2 = 0.05$, $k_2 = 0.02$ selected according to Lemma 4.1 renders the closed-loop system (4.5), (4.7), (4.28) GAS, as illustrated by numerical simulation in Figure 4.1. The plots of x_2 and u, along with the small values of λ_2 and k_2 illustrate the low-gain character of this design.

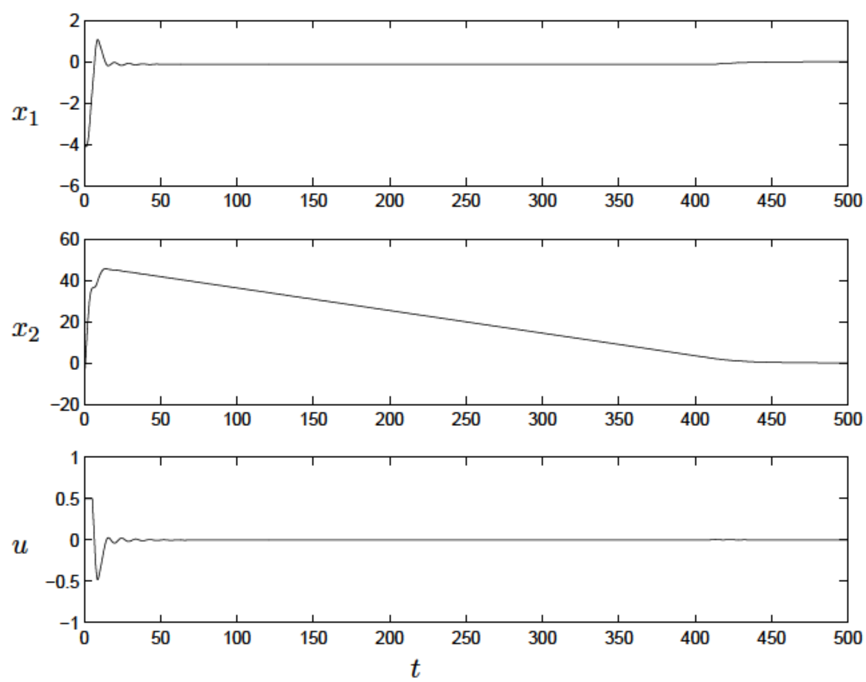

Figure 4.1: Closed-loop solutions with the redesigned control law (4.28).

4.3 Proof of Lemma 4.1

We give the proof for $i = 1$, that is, we show that if $\lambda_1 > 0$ and $k_1 > 0$ in (4.14) are selected sufficiently small, then the ξ_1-subsystem (4.15) with input u_1 and output v_1 satisfies Properties 1 and 2.

We prove Property 4.2 by using the asymptotic small-gain theorem of Teel [93]. To put the system (4.15) in a form suitable for the small-gain formulation we write

$$q(\xi, u, d) = A\xi + Bu + \tilde{q}(\xi, u, d), \tag{4.29}$$

where $\tilde{q}(\xi, u, 0) = o(\xi, u)$. From (4.15) and (4.29), the variable

$$z := x_1 - CA^{-1}\xi \tag{4.30}$$

is governed by

$$\dot{z} = (D - CA^{-1}B)u + G(\xi, u, d), \tag{4.31}$$

where $G(\xi, u, d) := G_1(\xi, u, d) - CA^{-1}\tilde{q}(\xi, u, d)$, thus $G(\xi, u, 0) = o(\xi, u)$. We substitute $\delta = D - CA^{-1}B > 0$ in (4.31), and write (4.15) as

$$\begin{aligned} \dot{z} &= \delta u + G(\xi, u, d) \\ \dot{\xi} &= q(\xi, u, d). \end{aligned} \tag{4.32}$$

From (4.14), we have

$$\delta u = -\delta\phi_1\left(k_1(x_1 - \frac{u_1}{k_1})\right). \tag{4.33}$$

It follows from the definition of saturation functions (1.56) that $\tilde{\phi}(s) := \delta\phi_1(s/\delta)$ is also a saturation function with level $\tilde{\lambda} := \delta\lambda_1$. Defining $\tilde{k} := \delta k_1$, we write (4.33) as

$$\delta u = -\tilde{\phi}\left(\tilde{k}(x_1 - \frac{u_1}{k_1})\right). \tag{4.34}$$

Substituting (4.34) in (4.32), and using (4.30), we obtain

$$\begin{aligned} \dot{z} &= -\tilde{\phi}\left(\tilde{k}(z + CA^{-1}\xi - \frac{u_1}{k_1})\right) + G(\xi, u, d) \\ \dot{\xi} &= q(\xi, u, d). \end{aligned} \tag{4.35}$$

Defining

$$y_1^{(2)} := CA^{-1}\xi, \quad y_2^{(2)} := \frac{G(\xi, u, d)}{\tilde{k}},$$

$$\nu^{(1)} := -\frac{u_1}{k_1}, \quad \nu^{(2)} := d, \quad y^{(1)} := z + y_1^{(2)} + \nu^{(1)},$$

we represent (4.35) as in Figure 4.2, where the subsystems $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are

$$\Sigma^{(1)}: \qquad \dot{z} = -\tilde{\phi}\left(\tilde{k}(z + y_1^{(2)} + \nu^{(1)})\right) + \tilde{k}y_2^{(2)} \tag{4.36}$$

$$\Sigma^{(2)}: \qquad \dot{\xi} = q(\xi, u, \nu^{(2)}), \quad u = -\phi_1(k_1 y^{(1)}). \tag{4.37}$$

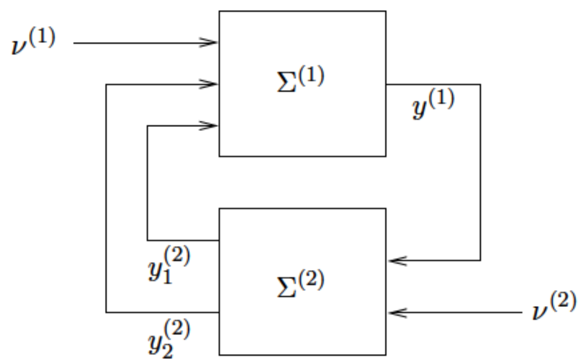

Figure 4.2: The feedback interconnection (4.36)-(4.37).

We now prove that if λ_1 and k_1 are sufficiently small, then the ξ_1-subsystem (4.15) with input (u_1, d), represented as in Figure 4.2, satisfies Property 4.2. We will equivalently show that if $\tilde{\lambda}$, $\tilde{k}$ are sufficiently small, then the composite output $(y^{(1)}, y^{(2)})$ of the feedback interconnection in Figure 4.2 with input $(\nu^{(1)}, \nu^{(2)})$ satisfies the asymptotic gain property

$$\|(y^{(1)}, y^{(2)})\|_a \leq \max\{c_{\nu,1}\,\|\nu^{(1)}\|_a\,,\; c_{\nu,2}\,\|\nu^{(2)}\|_a\}, \tag{4.38}$$

$c_{\nu,1}, c_{\nu,2} \geq 0$, for small $\|\nu^{(1)}\|_a$ and $\|\nu^{(2)}\|_a$. To prove (4.38), we rely on the following result adapted from Teel [93, Theorem 2].

Proposition 4.1 *For the feedback interconnection in Figure 4.2, suppose*

A1. For all locally bounded $\nu^{(1)}$, $\nu^{(2)}$, the solutions are unique and there are no finite escape times,

A2. There exist constants $\Delta_y > 0$, $c_1, c_2, c_3 \geq 0$ such that, if $\|y_2^{(2)}\|_a \leq \Delta_y$, then

$$\|y^{(1)}\|_a \leq \max\{\, c_1\,\|y_1^{(2)}\|_a\,,\; c_2\,\|y_2^{(2)}\|_a\,,\; c_3\,\|\nu^{(1)}\|_a\}, \tag{4.39}$$

A3. There exist continuous, nondecreasing functions $\gamma_1(\cdot), \gamma_2(\cdot)$ satisfying

$$\gamma_1(0) = \gamma_2(0) = 0\,,\; \gamma_1(\infty) < \infty,\; \gamma_2(\infty) < \infty \textit{ and } \gamma_2(\infty) \leq \Delta_y, \tag{4.40}$$

and there exist constants $\Delta_\nu, c_4, c_5 \geq 0$ such that, if $\|\nu^{(2)}\|_a \leq \Delta_\nu$, then

$$\|y_1^{(2)}\|_a \leq \max\{\gamma_1(\|y^{(1)}\|_a),\ c_4\,\|\nu^{(2)}\|_a\} \tag{4.41}$$

$$\|y_2^{(2)}\|_a \leq \max\{\gamma_2(\|y^{(1)}\|_a),\ c_5\,\|\nu^{(2)}\|_a\}, \tag{4.42}$$

A4. For all $s \neq 0$,

$$c_1\,\gamma_1(s) < s, \quad c_2\,\gamma_2(s) < s. \tag{4.43}$$

Then, there exist constants $c_{\nu,1}, c_{\nu,2}, \Delta_{\nu,2} \geq 0,\ \Delta_{\nu,1} > 0$ such that (4.38) holds for $\|\nu^{(1)}\|_a \leq \Delta_{\nu,1}$ and $\|\nu^{(2)}\|_a \leq \Delta_{\nu,2}$. □

We now prove that if λ_1 and k_1 are sufficiently small, then the conditions A1 through A4 of Proposition 4.1 are satisfied for $\Sigma^{(1)}$ and $\Sigma^{(2)}$ defined in (4.36) and (4.37).

Uniqueness of solutions follows from the Lipschitz continuity of the right hand sides of (4.36) and (4.37). The absence of finite escape times can be argued from the boundedness of u, as in Teel [93, Lemma 3.5].

To prove A2, we introduce the Lyapunov function $V(z) = z^2$ for (4.36), and note that $|y_2^{(2)}| < \tilde{\lambda}/2\tilde{k}$ and $|z| > |y_1^{(2)} + \nu^{(1)}| + |y_2^{(2)}|$ together imply $\dot{V} < 0$. Thus, if $\|y_2^{(2)}\|_a < \tilde{\lambda}/2\tilde{k} = \lambda_1/2k_1$, then

$$\|z\|_a \leq \|y_1^{(2)} + \nu^{(1)}\|_a + \|y_2^{(2)}\|_a. \tag{4.44}$$

It follows from (4.44) and $y^{(1)} := z + y_1^{(2)} + \nu^{(1)}$ that if $\|y_2^{(2)}\|_a < \lambda_1/2k_1$, then

$$\|y^{(1)}\|_a \leq 2\|y_1^{(2)}\|_a + \|y_2^{(2)}\|_a + 2\|\nu^{(1)}\|_a \leq \max\{6\|y_1^{(2)}\|_a, 6\|y_2^{(2)}\|_a, 6\|\nu^{(1)}\|_a\}. \tag{4.45}$$

We conclude from (4.45) that A2 holds with $c_1 = c_2 = c_3 = 6$ and $\Delta_y < \lambda_1/2k_1$, say,

$$\Delta_y = \frac{\lambda_1}{3k_1}. \tag{4.46}$$

To prove A3, we recall from Property 4.2 that, if $\|u\|_a \leq \Delta_u$, that is, $\lambda_1 \leq \Delta_u$ and, if $\|\nu^{(2)}\|_a \leq \Delta_d$, then

$$\|\xi\|_a \leq \max\{c_u\,\|u\|_a\,,\ c_d\,\|\nu^{(2)}\|_a\}. \tag{4.47}$$

From (1.56), we have $|u| = |\phi_1(k_1 y^{(1)})| \leq \min\{k_1|y^{(1)}|, \lambda_1\}$. Substituting in (4.47), $y_1^{(2)} := CA^{-1}\xi$ satisfies (4.41) with

$$\gamma_1(s) = |CA^{-1}|\,c_u \min\{k_1 s, \lambda_1\}, \quad c_4 = |CA^{-1}|c_d, \quad \Delta_\nu \leq \Delta_d. \tag{4.48}$$

It follows from (4.48) that $\gamma_1(\infty) = |CA^{-1}|\, c_u \lambda_1 < \infty$ and, for all $s \geq 0$,

$$\gamma_1(s) \leq k_1 c_u |CA^{-1}| s. \tag{4.49}$$

To show that (4.42) holds for $y_2^{(2)} := \frac{G(\xi,u,d)}{\delta k_1}$, we use Lipschitz continuity of $G(\xi, u, d)$ and note that, if ξ, u, d belong to a compact set, then $\exists L \geq 0$ such that

$$|G(\xi, u, d)| \leq |G(\xi, u, 0)| + L\,|d|. \tag{4.50}$$

By selecting Δ_ν to be finite, we restrict $\|d\|_a \leq \Delta_\nu$ to a compact interval. $\|u\|_a$ and $\|\xi\|_a$ also belong to a compact set because $|u| \leq \lambda_1$. Then, from $G(\xi, u, 0) = o(\xi, u)$, and the gain property (4.10), there exists $L_0 \geq 0$ and a nondecreasing, continuous function $\gamma_0(s) = o(s)$ such that

$$\|G(\xi, u, d)\|_a \leq \max\{\gamma_0(\|u\|_a), L_0\,\|d\|_a\}, \tag{4.51}$$

thus (4.42) is satisfied with

$$\gamma_2(s) = \frac{1}{k_1\delta}\gamma_0(\min\{k_1 s, \lambda_1\})\,, \quad c_5 = \frac{L_0}{k_1\delta}. \tag{4.52}$$

It is clear that $\gamma_2(\infty) < \infty$ as in (4.40). Comparing $\gamma_2(\infty) = \frac{1}{k_1\delta}\gamma_0(\lambda_1)$ to Δ_y in (4.46), and recalling $\gamma_0(s) = o(s)$, we conclude that $\gamma_2(\infty) \leq \Delta_y$ is satisfied if λ_1 is selected sufficiently small. From (4.52), a linear bound on $\gamma_2(s)$ is

$$\gamma_2(s) \leq \kappa s, \tag{4.53}$$

where $\kappa := \sup_{s\in[0,\frac{\lambda_1}{k_1}]} \frac{\gamma_2(s)}{s} = \sup_{s\in[0,\frac{\lambda_1}{k_1}]} \frac{\gamma_0(k_1 s)}{\delta\, k_1 s}$. Defining $\tilde{s} := k_1 s$, we get

$$\kappa = \sup_{\tilde{s}\in[0,\lambda_1]} \frac{\gamma_0(\tilde{s})}{\delta\,\tilde{s}}, \tag{4.54}$$

which is independent of k_1. Moreover, since $\gamma_0(\tilde{s}) = o(\tilde{s})$, κ can be rendered arbitrarily small by making λ_1 sufficiently small.

We prove A4 using the upper bounds on $\gamma_1(s)$ and $\gamma_2(s)$ given in (4.49) and (4.53), respectively. From (4.49), the first inequality in (4.43) is guaranteed by selecting k_1 such that

$$c_1 k_1 c_u |CA^{-1}| < 1\,. \tag{4.55}$$

From (4.53), the second inequality in (4.43) is guaranteed by selecting λ_1 and, hence, κ sufficiently small.

To show that Property 4.1 holds for (4.15), we note that the Jacobian linearization of $q_1(\xi_1, u_1, 0)$ is

$$A_1 = \begin{bmatrix} -k_1 D & C \\ -k_1 B & A \end{bmatrix}, \quad B_1 = \begin{bmatrix} D \\ B \end{bmatrix}. \tag{4.56}$$

Using $C_1 = [\,1\ 0_{1\times p}\,]$, $D_1 = 0$, and

$$A_1^{-1} = \begin{bmatrix} -\frac{1}{k_1} & \frac{1}{k_1} C A^{-1} \\ -A^{-1}B & A^{-1} + A^{-1}BCA^{-1} \end{bmatrix}, \tag{4.57}$$

we obtain

$$\delta_1 = D_1 - C_1 A_1^{-1} B_1 = \frac{1}{k_1}(D - CA^{-1}B) > 0\,. \tag{4.58}$$

With k_1 selected to satisfy the small-gain condition (4.55), A_1 is also guaranteed to be Hurwitz. This follows because the small gain analysis for (4.35) also holds with $G \equiv 0$, $u_1 \equiv 0$, $d \equiv 0$, and with $q(\xi, u) := q(\xi, u, 0)$ replaced by its linearization $A\xi + Bu$,

$$\begin{aligned} \dot{z} &= -\tilde{k}(z + CA^{-1}\xi) \\ \dot{\xi} &= A\xi + Bu, \quad u = -k_1(z + CA^{-1}\xi), \end{aligned} \tag{4.59}$$

which is $\dot{\xi}_1 = A_1 \xi_1$ expressed in the (z, ξ) coordinates. In Lemma 4.2 below, we show that the linearization

$$\dot{\xi} = A\xi + Bu, \tag{4.60}$$

has the same asymptotic input-to-state gain as the nonlinear system $\dot{\xi} = q(\xi, u)$. Then, since the small-gain condition (4.55) holds, the asymptotic small-gain theorem guarantees attractivity of the origin for $\dot{\xi}_1 = A_1 \xi_1$, which is equivalent to Hurwitz stability of A_1. □

Lemma 4.2 *Let $\dot{\xi} = q(\xi, u)$ be such that its Jacobian linearization (4.60) is Hurwitz, and for each locally bounded u satisfying $\|u\|_a \le \Delta_u$, $\Delta_u > 0$, and for each initial condition $\xi(0)$, the solution $\xi(t)$ exists for all $t \ge 0$ and satisfies*

$$\|\xi\|_a \le c_u \|u\|_a. \tag{4.61}$$

Then, for each locally bounded u, the solutions of (4.60) also satisfy (4.61).

Proof: Suppose, on the contrary, there exists a locally bounded $u^\star$ such that the solution ξ_L of the linearization (4.60) satisfies

$$\|\xi_L\|_a = c\|u^\star\|_a, \quad c > c_u. \tag{4.62}$$

Since (4.60) is a linear system, multiplying $u^\star$ by a constant does not change (4.62), thus we can set $\|u^\star\|_a = \epsilon \leq \Delta_u$. We rewrite $\dot{\xi} = q(\xi, u)$ as

$$\dot{\xi} = A\xi + Bu + \tilde{q}(\xi, u), \tag{4.63}$$

where $\tilde{q}(\xi, u) = o(\xi, u)$, and note that its solution with $u = u^\star$ can be expresses as $\xi(t) = \xi_L(t) + \xi_{\tilde{q}}(t)$, where $\xi_{\tilde{q}}$ is due to $\tilde{q}(\xi, u^\star)$, and,

$$\|\xi\|_a \geq |c\|u^\star\|_a - \|\xi_{\tilde{q}}\|_a| = \left|c - \frac{\|\xi_{\tilde{q}}\|_a}{\epsilon}\right| \|u^\star\|_a\,. \tag{4.64}$$

Because of (4.61), $\|\tilde{q}(\xi, u^\star)\|_a = o(\epsilon)$, and, hence, $\|\xi_{\tilde{q}}\|_a = o(\epsilon)$. Thus, for sufficiently small ϵ, $|c - \frac{\|\xi_{\tilde{q}}\|_a}{\epsilon}| > c_u$, which contradicts (4.61). □

4.4 Summary

We have presented a redesign of nested saturation control laws that makes them robust against input unmodeled dynamics, without any restrictions on their zero dynamics or relative degree. The achieved robustness property is due to the low-gain design, which is sufficient for stabilization of systems in feedforward form. Without the feedforward structure, global stabilization may not be possible in the presence of unmodeled dynamics, because low-gain designs may not be applicable.

Part II

Output-Feedback Designs

Chapter 5

A New Nonlinear Observer

The control designs we have presented in Part I assume that the full state of the plant is measured. We now remove this assumption and consider the problem where only a part of the state is available for measurement. In the rest of the dissertation we introduce new tools for nonlinear observer and observer-based control designs, and use them to design robust output-feedback control laws for systems with unmodeled dynamics.

In this chapter we introduce a new global observer design. The class of systems for which our observer is applicable are characterized by two restrictions which allow the observer error system to satisfy the *multivariable circle criterion.* First, a linear matrix inequality (LMI) is to be feasible, which implies a positive real property for the linear part of the observer error system. The second restriction is that the nonlinearities be nondecreasing functions of linear combinations of unmeasured states. This restriction ensures that the vector time-varying nonlinearity in the observer error system satisfies the sector condition of the circle criterion. The observer design in Section 5.1 is accompanied by its reduced-order variant in Section 5.2.

The proposed observer design is constructive in the sense that the issues of existence and the evaluation of observer matrices satisfying the circle criterion are resolved by efficient LMI computations. A further advantage of our design is its robustness against inexact modeling of nonlinearities. This robustness property is analyzed in Section 5.3, and bounds are given within which the observer error gradually increases with an increase in the modeling error.

5.1 The New Observer Design

We now present the observer design for the control system

$$\begin{aligned} \dot{x} &= Ax + G\gamma(Hx) + \varrho(y,u) \\ y &= Cx\,, \end{aligned} \tag{5.1}$$

where $x \in I\!R^n$ is the state, $y \in I\!R^p$ is the measured output, $u \in I\!R^m$ is the control input, the pair (A, C) is detectable, and, $\gamma(\cdot)$ and $\varrho(\cdot,\cdot)$ are locally Lipschitz. The state-dependent nonlinearity $\gamma(Hx)$ is an r-dimensional vector where each entry is a function of a linear combination of the states

$$\gamma_i = \gamma_i(\sum_{j=1}^{n} H_{ij}x_j), \quad i = 1, \cdots, r\,. \tag{5.2}$$

The main restriction is that each $\gamma_i(\cdot)$ is nondecreasing, that is, for all $a, b \in I\!R$, it satisfies

$$(a-b)[\gamma_i(a) - \gamma_i(b)] \geq 0\,. \tag{5.3}$$

If $\gamma_i(\cdot)$ is continuously differentiable, then (5.3) means that $d\gamma_i(v)/dv \in [0, \infty)$ for all $v \in I\!R$. If, instead, $\gamma_i(\cdot)$ satisfies $d\gamma_i(v)/dv \in [g_i, \infty)$ with $g_i \neq 0$, we can still represent the system as in (5.1)-(5.3) by defining a new function $\tilde{\gamma}_i(v) := \gamma_i(v) - g_i v$ which satisfies $d\tilde{\gamma}_i(v)/dv \in [0, \infty)$, and absorbing $g_i v$ in the linear part of the system.

The observer for system (5.1)-(5.3) is

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + G\,\gamma(H\hat{x} + K(C\hat{x} - y)) + \varrho(y, u)\,, \tag{5.4}$$

and the design task is to determine the observer matrices $K \in I\!R^{r\times p}$ and $L \in I\!R^{n\times p}$. Note that the nonlinear injection $\gamma(H\hat{x} + K(C\hat{x} - y))$ is analogous to w^3 in the van der Pol example in Section 1.2.4. The same observer design can be applied to the system (5.1) when the nonlinearity $\gamma(Hx)$ also depends on y and u. In this case we require that the nondecreasing property (5.3) hold for each $y \in I\!R^p$ and $u \in I\!R^m$.

For the observer equation to be defined, the uniqueness of the solutions $x(t)$ of (5.1) is guaranteed by restricting the control law $u = \alpha(y, \hat{x}, t)$ to be locally Lipschitz in $(y, \hat{x})$, uniformly in t. As will be further clarified in Chapter 7, $u = \alpha(y, \hat{x}, t)$ is also assumed to ensure that $x(t)$ does not escape to infinity in finite time, that is $x(t) \in \mathcal{L}_\infty^e$.

From (5.1) and (5.4), the dynamics of the observer error $e = x - \hat{x}$ are governed by

$$\dot{e} = (A + LC)e + G\,[\gamma(v) - \gamma(w)]\,, \tag{5.5}$$

where

$$v := Hx \quad \text{and} \quad w := H\hat{x} + K(C\hat{x} - y)\,. \tag{5.6}$$

We begin the observer design by representing the observer error system (5.5) as the feedback interconnection of a linear system and a multivariable sector nonlinearity. To this end, we introduce a new variable

$$z := v - w = (H + KC)e\,, \tag{5.7}$$

and view $\gamma(v) - \gamma(w)$ as a function of t and z

$$\varphi(t,z) := \gamma(v) - \gamma(w)\,, \tag{5.8}$$

where the time dependence of $\varphi(t,z)$ is due to $v(t) = Hx(t)$. Substituting (5.8), we rewrite the observer error system (5.5) as

$$\begin{aligned} \dot{e} &= (A + LC)e + G\varphi(t,z) \\ z &= (H + KC)e\,, \end{aligned} \tag{5.9}$$

and note from (5.3) that each component of $\varphi(t,z)$ satisfies

$$z_i \varphi_i(t, z_i) \geq 0\,, \quad \forall z_i \in I\!R\,. \tag{5.10}$$

Thanks to this sector property, we can employ the multivariable circle criterion and derive a condition that guarantees the exponential convergence of the observer error $e(t)$ to zero.

Theorem 5.1 *Consider the plant (5.1), observer (5.4), and suppose $x(t)$ exists for all $t \geq 0$. If there exists a matrix $P = P^T > 0$, a constant $\nu > 0$, and a diagonal matrix $\Lambda > 0$ such that*

$$\begin{bmatrix} (A+LC)^T P + P(A+LC) + \nu I & PG + (H+KC)^T \Lambda \\ G^T P + \Lambda(H+KC) & 0 \end{bmatrix} \leq 0, \tag{5.11}$$

then the observer error $e(t)$ satisfies for all $t \geq 0$

$$|e(t)| \leq \kappa |e(0)| \exp(-\beta t)\,, \tag{5.12}$$

where $\kappa = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, $\beta = \frac{\nu}{2\lambda_{max}(P)}$.

Proof: From (5.9), the time derivative of $V(e) = e^T P e$ is

$$\dot{V} = e^T[(A + LC)^T P + P(A + LC)]e + 2e^T PG\varphi(t, z)\,, \tag{5.13}$$

and, in view of (5.11),

$$\dot{V} \leq -\nu|e|^2 - 2e^T(H + KC)^T \Lambda\varphi(t, z)\,. \tag{5.14}$$

Substituting $(H + KC)e = z$, we rewrite (5.14) as

$$\dot{V} = -\nu|e|^2 - 2\sum_{i=1}^{r} \lambda_i z_i \varphi_i(t, z_i)\,, \tag{5.15}$$

where $\lambda_i > 0$, $i = 1, \cdots, r$, are the entries of the diagonal matrix Λ. Then, the sector property (5.10) yields

$$\dot{V} \leq -\nu|e|^2\,, \tag{5.16}$$

from which (5.12) follows. $\square$

With Theorem 5.1, the observer design for system (5.1) is reduced to the problem of finding observer matrices K and L such that (5.11) is satisfied with some $P = P^T > 0$, $\Lambda > 0$, and $\nu > 0$. The existence of such K and L depends on A, C, G and H, and cannot be ascertained *a priori*. However, (5.11) is a LMI in P, PL, Λ, ΛK and ν. Therefore, we can use the efficient numerical tools available for LMI's to determine whether the problem is feasible and, if so, to compute K and L.

We have gained additional design freedom in the LMI (5.11) by introducing $\Lambda = diag(\lambda_1, \cdots, \lambda_r)$ as a parameter instead of $\Lambda = I$. Due to the special structure of $\varphi(t, z)$, in which every individual entry $\varphi_i(t, z_i)$ satisfies the sector condition $z_i\varphi_i(t, z_i) \geq 0$, the sum $\sum_{i=1}^{r} \lambda_i z_i \varphi_i(t, z_i) = \varphi(t, z)^T \Lambda z$ is nonnegative for any choice of $\lambda_i > 0$, $i = 1, \cdots, r$. This means that the framed block in the feedback loop in Figure 5.1 below is a multivariable sector nonlinearity for any diagonal $\Lambda > 0$. Thus, from the multivariable circle criterion, asymptotic stability of the closed-loop is guaranteed if the linear system with input ϑ and output Λz is SPR. Indeed, the LMI (5.11) constitutes the required SPR condition.

If the LMI (5.11) holds with $\nu = 0$, that is, if the linear system in Figure 5.1 with input ϑ and output Λz is PR but not SPR, then (5.16) guarantees $|e(t)| \leq \kappa|e(0)|$, but not necessarily $e(t) \to 0$. Under the additional assumption that $x(t) \in \mathcal{L}_\infty$, we now prove that $e(t)$ still converges to zero exponentially.

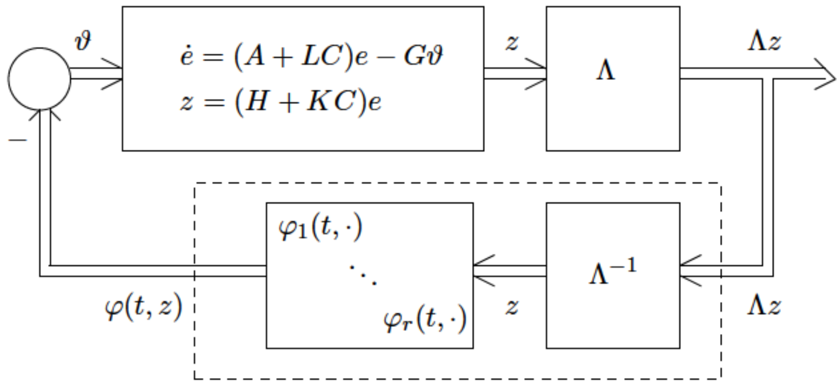

Figure 5.1: Observer error system.

Theorem 5.2 *Consider the plant (5.1) and observer (5.4). If $P = P^T > 0$ satisfies (5.11) with $\nu = 0$, and $A + LC$ is Hurwitz, then for any compact set $\mathcal{C} \subset \mathbb{R}^{2n}$, with $(x(t), \hat{x}(t)) \in \mathcal{C}$ for all $t \geq 0$, there exist positive constants $\kappa_{\mathcal{C}}$ and $\beta_{\mathcal{C}}$ such that*

$$|e(t)| \leq \kappa_{\mathcal{C}} |e(0)| \exp(-\beta_{\mathcal{C}} t), \quad \forall t \geq 0. \tag{5.17}$$

Proof: We will use the following lemma proved by Teel [94]:

Lemma 5.1 *If there exist positive constants θ_1 and θ_2 such that, for each $t_0 \geq 0$, and for all $t \geq t_0$,*

$$|e(t)| \leq \theta_1 |e(t_0)| \tag{5.18}$$

$$\int_{t_0}^{t} |e(\tau)|^2 d\tau \leq \theta_2 |e(t_0)|^2, \tag{5.19}$$

then $e(t)$ satisfies (5.17) with constants $\kappa_{\mathcal{C}}, \beta_{\mathcal{C}} > 0$ that depend on θ_1, θ_2.

The first inequality (5.18) holds because $\dot{V} \leq 0$ by (5.16). To prove (5.19), we first note that the observer error system (5.9) driven by $\varphi(t, z)$ is $\mathcal{L}_2$-stable, because $A+LC$ is Hurwitz. This means that there exist $b_1 > 0$ and $c > 0$ such that

$$\int_{t_0}^{t} |e(\tau)|^2 d\tau \leq b_1 |e(t_0)|^2 + c \int_{t_0}^{t} |\varphi(\tau, z(\tau))|^2 d\tau. \tag{5.20}$$

To prove that (5.19) holds, we show that

$$\int_{t_0}^{t} |\varphi(\tau, z(\tau))|^2 d\tau \leq b_2 |e(t_0)|^2. \tag{5.21}$$

Since $(x(t), \hat{x}(t)) \in \mathcal{C}$ for all $t \geq 0$, the components v_i, w_i of v and w, defined in (5.6), remain for all $t \geq 0$ in a compact interval $\mathcal{C}_i$, in which

$$d_i := \sup_{v_i, w_i \in \mathcal{C}_i,\, v_i \neq w_i} \frac{\gamma_i(v_i) - \gamma_i(w_i)}{v_i - w_i} \tag{5.22}$$

is finite, because $\gamma_i(\cdot)$ is locally Lipschitz. It follows from (5.3), (5.22), and $z_i = v_i - w_i$ that whenever $z_i \neq 0$, the inequality

$$0 \leq \frac{\varphi_i(t, z_i)}{z_i} \leq d_i \tag{5.23}$$

holds for all $t \geq 0$. Multiplying the right inequality in (5.23) by $\frac{1}{d_i} z_i \varphi(t, z_i)$, which is nonnegative by the left inequality, we obtain

$$\frac{1}{d_i} \varphi_i(t, z_i)^2 \leq z_i \varphi_i(t, z_i)\,, \tag{5.24}$$

and, in view of (5.15),

$$\frac{1}{2}\dot{V} \leq -\sum_{i=1}^{r} \lambda_i z_i \varphi_i(t, z_i) \leq -\sum_{i=1}^{r} \frac{\lambda_i}{d_i} \varphi_i(t, z_i)^2 \leq -b\,|\varphi(t, z)|^2\,, \tag{5.25}$$

where $b := \min_i \{\frac{\lambda_i}{d_i}\} > 0$. Integrating both sides of (5.25) from t_0 to t, we verify that (5.21) holds with $b_2 = \frac{\lambda_{max}(P)}{2b}$. Substituting (5.21) in (5.20), we see that (5.19) holds with $\theta_2 = b_1 + c b_2$. $\square$

We now illustrate the use of Theorem 5.2 in a situation where the LMI (5.11) is not feasible for $\nu > 0$.

Example 5.1 *The system*

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 \\ \dot{x}_2 &= x_2 + x_3 - \exp(x_2) + u \\ \dot{x}_3 &= 2u \\ y &= x_1 \end{aligned} \tag{5.26}$$

is of the form (5.1) with $\varrho(y, u) = [\,y^2 \;\; u \;\; 2u\,]^T$,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = [\,1 \;\; 0 \;\; 0\,]\,, \quad G = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad H = [\,0 \;\; 1 \;\; 0\,]\,.$$

Since the nonlinearity $\gamma(Hx) = \gamma_1(x_2) = \exp(x_2)$ is nondecreasing as in (5.3), we proceed with the observer design. With A, C, G and H as above, the LMI (5.11) is not feasible with $\nu > 0$. However, with $\nu = 0$, it is feasible and a solution is

$$P = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -2 \\ -4 \\ -1 \end{bmatrix}, \quad K = -1\,, \quad \Lambda = 1\,. \tag{5.27}$$

Because $A+LC$ is Hurwitz, we conclude from Theorem 5.2 that the resulting observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - 2(\hat{x}_1 - y) + y^2 \\ \dot{\hat{x}}_2 &= \hat{x}_2 + \hat{x}_3 - 4(\hat{x}_1 - y) - \exp(\hat{x}_2 - (\hat{x}_1 - y)) + u \\ \dot{\hat{x}}_3 &= -(\hat{x}_1 - y) + 2u \end{aligned} \tag{5.28}$$

guarantees $e(t) \to 0$ as $t \to \infty$, if a control law can be designed to ensure $x(t) \in \mathcal{L}_\infty$.

5.2 Reduced-Order Observer

In applications it may be more convenient to employ a reduced-order observer, which generates estimates only for the unmeasured states. The design of such an observer starts with a preliminary change of coordinates such that the output y consists of the first p entries of the state vector $x = [\,y^T \ x_o^T\,]^T$. In the new coordinates, the system (5.1) is

$$\begin{aligned} \dot{y} &= A_1 x_o + G_1\gamma(H_1 y + H_2 x_o) + \varrho_1(y,u) \\ \dot{x}_o &= A_2 x_o + G_2\gamma(H_1 y + H_2 x_o) + \varrho_2(y,u)\,, \end{aligned} \tag{5.29}$$

where the linear terms in y are incorporated in $\varrho_1(y,u)$ and $\varrho_2(y,u)$. An estimate of x_o will be obtained via $\chi := x_o + Ny$, where $N \in I\!R^{(n-p)\times p}$ is to be designed. From (5.29), the derivative of χ is:

$$\dot{\chi} = (A_2 + NA_1)\chi + (G_2 + NG_1)\gamma(H_2\chi + (H_1 - H_2N)y) + \bar{\varrho}(y,u)\,, \tag{5.30}$$

where $\bar{\varrho}(y,u) := N\varrho_1(y,u) + \varrho_2(y,u) - (A_2 + NA_1)Ny$. In this χ-subsystem, the output injection matrix N has altered the A_2 and G_2 matrices of the x_o-subsystem (5.29).

To obtain the estimate

$$\hat{x}_o = \hat{\chi} - Ny\,, \tag{5.31}$$

we employ the observer

$$\dot{\hat{\chi}} = (A_2 + NA_1)\hat{\chi} + (G_2 + NG_1)\gamma(H_2\hat{\chi} + (H_1 - H_2N)y) + \bar{\varrho}(y,u)\,. \tag{5.32}$$

From (5.30) and (5.32), the dynamics of $e_o := x_o - \hat{x}_o = \chi - \hat{\chi}$ are governed by

$$\dot{e}_o = (A_2 + NA_1)e_o + (G_2 + NG_1)[\gamma(v_o) - \gamma(w_o)]\,, \tag{5.33}$$

where $v_o := H_2\chi + (H_1 - H_2N)y$ and $w_o := H_2\hat{\chi} + (H_1 - H_2N)y$. We let $z := v_o - w_o = H_2e_o$, and denote $\varphi(t,z) = \gamma(v_o) - \gamma(w_o)$. Then, the nondecreasing property (5.3) implies that $z_i\varphi_i(t,z_i) \geq 0$ for all $i = 1,\cdots,r$. Derivations similar to those in Section 5.1 yield the following LMI in P_o, P_oN, ν and Λ:

$$\left[\begin{array}{cc} (A_2 + NA_1)^TP_o + P_o(A_2 + NA_1) + \nu I & P_o(G_2 + NG_1) + H_2^T\Lambda \\ (G_2 + NG_1)P_o + \Lambda H_2 & 0 \end{array}\right] \leq 0\,. \tag{5.34}$$

If this LMI is satisfied with a matrix $P_o = P_o^T > 0$, a constant $\nu \geq 0$, and a diagonal matrix $\Lambda > 0$, then it is not difficult to show that, with appropriate modifications, Theorems 5.1 and 5.2 hold for the observer error $e_o(t)$.

To illustrate the analog of Theorem 5.2, we design a reduced-order observer for the system (5.26).

Example 5.2 *The system (5.26) is of the form (5.29) with* $x_o = [x_2\ x_3]^T$, $\varrho_1(y,u) = y^2$, $\varrho_2(y,u) = [u\ 2u]^T$,

$$A_1 = [\,1 \quad 0\,]\,,\quad A_2 = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right],\quad G_1 = 0,\quad G_2 = \left[\begin{array}{c} -1 \\ 0 \end{array}\right],\quad H_1 = 0,\quad H_2 = [\,1 \quad 0\,]\,. \tag{5.35}$$

The LMI (5.34) is feasible with $\nu = 0$, *and a solution is*

$$P_o = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right],\quad N = \left[\begin{array}{c} -2 \\ -1 \end{array}\right],\quad \Lambda = 1\,. \tag{5.36}$$

Since $A_2 + NA_1$ *is Hurwitz, the resulting observer*

$$\begin{array}{rcl} \dot{\hat{\chi}}_2 & = & -\hat{\chi}_2 + \hat{\chi}_3 - \exp(\hat{\chi}_2 + 2y) + (-2y^2 - y + u) \\ \dot{\hat{\chi}}_3 & = & -\hat{\chi}_2 + (-y^2 - 2y + 2u) \\ \hat{x}_2 & = & \hat{\chi}_2 + 2y \\ \hat{x}_3 & = & \hat{\chi}_3 + y \end{array} \tag{5.37}$$

guarantees $(e_2(t),\, e_3(t)) \to 0$ *as* $t \to \infty$, *if a control law can be designed to ensure* $x(t) \in \mathcal{L}_\infty$. □

5.3 Robustness Against Inexact Modeling of Nonlinearities

Thus far we have assumed exact knowledge of the system nonlinearities. To analyze the effects of modeling errors, we suppose that instead of (5.1), the system is

$$\dot{x} = Ax + G[\gamma(Hx) + \Delta(Hx)\mu(t)] + \varrho(y,u)\,, \tag{5.38}$$

where $\mu(t)$ is a bounded disturbance. Then, the nominal observer (5.4) yields the observer error system

$$\dot{e} = (A + LC)e + G\left[\gamma(v) - \gamma(w) + \Delta(v)\mu(t)\right], \tag{5.39}$$

where $v = Hx$, $w = H\hat{x} + K(C\hat{x} - y)$.

In Theorem 5.3 below, we characterize nonlinearities $\Delta(\cdot)$ for which the observer (5.4) guarantees an ISS property from the disturbance $\mu(t)$ to the observer error $e(t)$.

Theorem 5.3 *Consider the plant (5.38) and the observer (5.4). Suppose $x(t)$ exists for all $t \geq 0$, and that the LMI (5.11) holds with a matrix $P = P^T > 0$, a constant $\nu > 0$, and a diagonal matrix $\Lambda > 0$. If, for each $i = 1, \cdots, r$, there exists a class-$\mathcal{K}$ function $\sigma_i(\cdot)$ such that*

$$(a-b)[\gamma_i(a) - \gamma_i(b) + \Delta_i(a)\mu] \geq -\sigma_i(|\mu|) \quad \forall a, b, \mu \in \mathbb{R}\,, \tag{5.40}$$

then the observer error $e(t)$ satisfies, for all $t \geq 0$,

$$|e(t)| \leq \kappa |e(0)| \exp(-\beta t) + \rho\left(\sup_{0\leq\tau\leq t} |\mu(\tau)|\right), \tag{5.41}$$

where $\kappa = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, $\beta = \frac{\nu}{2\lambda_{max}(P)}$, and the ISS-gain from $\mu(t)$ to $e(t)$ is

$$\rho(\cdot) = \kappa\sqrt{\frac{2}{\nu}\sum_{i=1}^{r}\lambda_i\,\sigma_i(\cdot)}. \tag{5.42}$$

Proof: We use $V = e^T Pe$ as an ISS-Lyapunov function, and evaluate its derivative for (5.39):

$$\dot{V} \leq -\nu|e|^2 - 2\sum_{i=1}^{r}\lambda_i\,(v_i - w_i)\left[\gamma_i(v_i) - \gamma_i(w_i) + \Delta_i(v_i)\mu\right]. \tag{5.43}$$

Substituting (5.40), we obtain

$$\dot{V} \leq -\nu|e|^2 + 2\sum_{i=1}^{r} \lambda_i \, \sigma_i(|\mu|) \leq -2\beta V + 2\sum_{i=1}^{r} \lambda_i \, \sigma_i(|\mu|) \,, \tag{5.44}$$

from which it follows that

$$V(t) \leq V(0)\exp(-2\beta t) + \frac{1}{\beta}\left(\sum_{i=1}^{r} \lambda_i \sup_{0\leq\tau\leq t} \sigma_i(|\mu(\tau)|)\right) . \tag{5.45}$$

This yields

$$|e(t)| \leq \kappa|e(0)|\exp(-\beta t) + \sqrt{\frac{1}{\beta\lambda_{min}(P)}\left(\sum_{i=1}^{r} \lambda_i \sup_{0\leq\tau\leq t} \sigma_i(|\mu(\tau)|)\right)} \tag{5.46}$$

and, (5.41) and (5.42) are obtained by substituting $\frac{1}{\beta\lambda_{min}(P)} = \frac{2\kappa^2}{\nu}$. □

The ISS property established by Theorem 5.3 shows that $e(t)$ degrades gracefully with the increase in the magnitude of the disturbance $\mu(t)$. As $\mu(t)$ vanishes, we recover the convergence result of Theorem 5.1. The dependence of admissible nonlinearities $\Delta(\cdot)$ on $\gamma(\cdot)$ is characterized by (5.40). For example, if $\gamma(\cdot)$ is cubic, then $\Delta(\cdot)$ is allowed to be linear. In this case, (5.40) is satisfied because

$$(a-b)[a^3 - b^3 + a\mu] \geq -\frac{1}{3}\mu^2 \tag{5.47}$$

holds for all $a, b, \mu \in I\!R$, due to the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. On the other hand, (5.40) does not hold for cubic $\gamma(\cdot)$ and quadratic $\Delta(\cdot)$. To see this, we evaluate $(a-b)[a^3 - b^3 + a^2\mu]$ with $b = a + \frac{1}{a}$, and note that, for any fixed $\mu > 0$, the resulting function $3 + \frac{3}{a^2} + \frac{1}{a^4} - a\mu$ tends to $-\infty$ as $a \to +\infty$, thus violating (5.40).

It is not difficult to prove the analog of Theorem 5.3 for the reduced-order observer (5.32), in which case $e(t)$ is to be replaced by $e_o(t)$.

5.4 Summary

An observer design is presented for systems with monotone nonlinearities in the unmeasured states. The new design represents the observer error system as the feedback interconnection of a linear system and a multivariable sector nonlinearity. The issues of existence and the evaluation of the observer matrices K and L satisfying the circle criterion are resolved by efficient LMI computations. The robustness of the new observer to inexact modeling of nonlinearities is analyzed, and ISS bounds are derived within which the observer error increases with an increase in the modeling error.

Chapter 6

Feasibility Conditions for the Observer Design

Thus far, the feasibility of our observer design was left to be resolved by iterative LMI computations. In this chapter we derive structural conditions that characterize feasibility. We first show that rendering a linear system SPR by output injection is equivalent to rendering its dual system SPR by control. Then, we characterize the feasibility of this dual problem, and use it to establish necessary and sufficient feasibility conditions for the observer design.

The duality property is discussed in Section 6.1. The feasibility of the dual problem is studied in Section 6.2. In Section 6.3, we give the feasibility conditions for the observer design, and illustrate them on examples. Finally, in Section 6.4, we prove that the feasibility conditions for the reduced-order observer are the same as those for the full-order observer. The lengthy proofs are given in Section 6.5.

6.1 The Dual Problem

The observer design in the previous chapter relied on solving the LMI (5.11), rewritten here as

$$(A + LC)^T P + P(A + LC) < 0 \tag{6.1}$$

$$PG + (H + KC)^T \Lambda = 0. \tag{6.2}$$

Multiplying the inequality (6.1) from both sides by $\mathcal{P} := P^{-1}$, and multiplying the equality (6.2) from the left by $\mathcal{P}$, and from the right by $\Omega := \Lambda^{-1}$, we obtain

$$\mathcal{P}(\mathcal{A} + \mathcal{B}\mathcal{F}) + (\mathcal{A} + \mathcal{B}\mathcal{F})^T \mathcal{P} < 0 \tag{6.3}$$

$$\mathcal{P}(\mathcal{G} + \mathcal{B}\mathcal{E}) + \mathcal{H}^T \Omega = 0, \tag{6.4}$$

where

$$\mathcal{A} := A^T, \quad \mathcal{B} := C^T, \quad \mathcal{E} := K^T, \quad \mathcal{F} := L^T, \quad \mathcal{G} := H^T, \quad \mathcal{H} := G^T. \tag{6.5}$$

In view of the PR Lemma, (6.3) and (6.4) mean that the control system

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + \mathcal{G}w \tag{6.6}$$

$$z = -\mathcal{H}x \tag{6.7}$$

is rendered SPR from the disturbance w to the output Ωz by the control law

$$u = \mathcal{F}x + \mathcal{E}w. \tag{6.8}$$

Thus, our observer design is dual to a control problem in which the system (6.6)-(6.7) is to be rendered SPR by state feedback and disturbance feedforward. The state feedback matrix $\mathcal{F}$ is the dual of the observer gain matrix L, and the feedforward matrix $\mathcal{E}$ is the dual of the nonlinear injection matrix K.

This dual problem encompasses the *circle criterion control design* of Janković *et al.* [30]. For the system in Figure 6.1, where the nonlinearity $w = -\phi(t, z)$ satisfies the sector property $z\phi(t, z) \geq 0$, the design in [30] renders the linear system SPR with respect to the input w by using a control law of the form (6.8), thus achieving global uniform asymptotic stability via the circle criterion. This means that $\mathcal{F}$ and $\mathcal{E}$ are to be designed such that (6.3) and (6.4) hold for some $\mathcal{P} = \mathcal{P}^T > 0$.

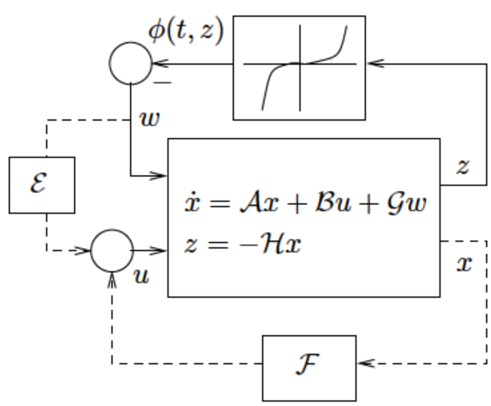

Figure 6.1: Circle criterion control design.

In [30], the feasibility issue, that is, the existence conditions of a control law of the form (6.8) that renders the resulting closed-loop system SPR from w to z, is left to be resolved by iterative LMI computations. None of the feedback passivation conditions in the literature are applicable to the feasibility problem for the design in Figure 6.1. The results such as Fradkov [16], and Kokotović and Sussmann [47], deal with 'direct' passivation for the control input u, and not with 'indirect' passivation for the disturbance input w. Recent indirect passivation conditions derived in the H_∞-framework by Safonov *et al.* [80], Haddad and Bernstein [22], Sun *et al.* [90], and Turan *et al.* [100] assume that the relative degree from w to z is zero, which does not hold for (6.6)-(6.7).

6.2 Feasibility of the Circle Criterion Design

In this section we derive structural conditions that completely characterize the feasibility of the circle criterion design. This result will be used in the next section to establish feasibility conditions for the observer design. To simplify the derivations, we restrict our analysis to a single nonlinearity, that is, Ω and $\mathcal{E}$ in (6.3)-(6.4) are scalars. We let $\Omega = 1$, $\mathcal{E} = \rho$, and analyze the existence of $\mathcal{F}$, ρ and $\mathcal{P} = \mathcal{P}^T > 0$ satisfying

$$(\mathcal{A} + \mathcal{B}\mathcal{F})^T\mathcal{P} + \mathcal{P}(\mathcal{A} + \mathcal{B}\mathcal{F}) < 0 \tag{6.9}$$

$$\mathcal{P}(\mathcal{G} + \rho\mathcal{B}) + \mathcal{H}^T = 0. \tag{6.10}$$

The case where ρ is constrained to be zero is of separate interest, because then the control law (6.8) can be implemented without the exact knowledge of the nonlinearity $w = -\phi(t, z)$.

After a change of coordinates and a preliminary state feedback, the system (6.6)-(6.7) is represented as

$$\dot{\xi} = A_0\xi + E_0 y_1 + G_0 w \tag{6.11}$$

$$\begin{aligned} \dot{y}_1 &= y_2 + g_1 w \\ \dot{y}_2 &= y_3 + g_2 w \\ &\vdots \\ \dot{y}_r &= u + g_r w \end{aligned} \tag{6.12}$$

$$z = y_1, \tag{6.13}$$

where $\xi \in I\!R^{n-r}$, and r is the relative degree from the output z to the control input u. As will be shown later, a crucial ingredient of the feasibility problem is the existence of a matrix $Y = Y^T > 0$ satisfying

$$A_0Y + YA_0^T + 2E_0G_0^T + 2G_0E_0^T < 0, \tag{6.14}$$

or, equivalently,

$$A_0 Y + Y A_0^T < -(E_0 + G_0)(E_0 + G_0)^T + (E_0 - G_0)(E_0 - G_0)^T.$$

When A_0 is Hurwitz, the existence of Y is immediate. If A_0 is not Hurwitz, we decompose the ξ-subsystem (6.11) into three subsystems,

$$\dot{\xi}_i = A_0^i \xi_i + E_0^i x_1 + G_0^i w, \quad i = 1, 2, 3, \tag{6.15}$$

such that $\sigma(A_0^1) \subset \mathbb{C}^+$, $\sigma(A_0^2) \subset \mathbb{C}^0$, $\sigma(A_0^3) \subset \mathbb{C}^-$, and consider $U = U^T$, $V = V^T$ defined by

$$A_0^1 U + U A_0^{1T} = (E_0^1 - G_0^1)(E_0^1 - G_0^1)^T \tag{6.16}$$

$$A_0^1 V + V A_0^{1T} = (E_0^1 + G_0^1)(E_0^1 + G_0^1)^T. \tag{6.17}$$

Theorem 6.1 ($\rho = 0$) *A state feedback control law $u = \mathcal{F}x$ that renders (6.11)-(6.13) SPR from w to $z = y_1$ exists if and only if*

$$g_1 > 0, \quad g_2 < 0, \quad U - V > \frac{2}{g_1} G_0^1 G_0^{1T}, \tag{6.18}$$

and

$$w^*(E_0^2 - G_0^2)(E_0^2 - G_0^2)^T w > w^*(E_0^2 + G_0^2)(E_0^2 + G_0^2)^T w \tag{6.19}$$

for every eigenvector w of A_0^{2T}. □

The proof is given in Section 6.5.

As a corollary, we give the feasibility conditions for the case $\rho \neq 0$, when the control law $u = \mathcal{F}x + \rho w$ can arbitrarily assign $\tilde{g}_r = g_r + \rho$ in (6.12). If $r \geq 3$, the feasibility conditions for $\rho \neq 0$ are the same as in Theorem 6.1, because they do not depend on g_r. However, if $r = 2$, then $g_2 < 0$ is not required because we can use ρ to satisfy $\tilde{g}_2 < 0$. Likewise, if $r = 1$ then $g_1 > 0$ is no longer required. Moreover, since $\tilde{g}_1 > 0$ can be arbitrarily large, $U - V > \frac{2}{g_1} G_0^1 G_0^{1T}$ is replaced by the less restrictive condition $U > V$.

Corollary 6.1 ($\rho \neq 0$) *When $r = 1$, a control law $u = \mathcal{F}x + \rho w$ that renders (6.11)-(6.13) SPR from w to $z = y_1$ exists if and only if $U > V$ and (6.19) holds for every eigenvector w of A_0^{2T}. When $r = 2$, $g_1 > 0$ and $U - V > \frac{2}{g_1} G_0^1 G_0^{1T}$ are required in addition. When $r \geq 3$, all the conditions of Theorem 6.1 are required.* □

6.3 Feasibility of the Observer Design

We now derive necessary and sufficient conditions for the feasibility of the observer design. For the system

$$\begin{aligned} \dot{x} &= Ax + G\gamma(Hx) + \varrho(y,u) \\ y &= Cx\,, \end{aligned} \tag{6.20}$$

with a single output $y \in I\!R$, and a single nondecreasing nonlinearity $\gamma(\cdot)$, we determine when the LMI

$$\begin{bmatrix} (A+LC)^T P + P(A+LC) + \nu I & PG + (H+KC)^T \\ G^T P + (H+KC) & 0 \end{bmatrix} \le 0, \tag{6.21}$$

is feasible. When $\nu = 0$ is allowed as in Theorem 5.2, the feasibility conditions become cumbersome, therefore we restrict our discussion to feasibility with $\nu > 0$.

We omit $\varrho(y,u)$ and the linear terms in y from the right-hand side of (5.1) because they do not affect feasibility, and represent the resulting system in the following canonical form

$$\begin{aligned} y &= y_1 \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\cdots \\ \dot{y}_r &= \Pi\xi - \gamma(\Sigma\xi + \sigma_1 y_1 + \cdots + \sigma_r y_r) \\ \dot{\xi} &= S\xi, \end{aligned} \tag{6.22}$$

where r is the relative degree from the output y to the nonlinearity $\gamma(\cdot)$. Next, we decompose S, Π, and Σ as

$$S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}, \quad \Pi = [\,\Pi_1\ \Pi_2\ \Pi_3\,], \quad \Sigma = [\,\Sigma_1\ \Sigma_2\ \Sigma_3\,], \tag{6.23}$$

where $\sigma(S_1) \subset \mathbb{C}^+$, $\sigma(S_2) \subset \mathbb{C}^0$, $\sigma(S_3) \subset \mathbb{C}^-$, and define $U = U^T$, $V = V^T$ by

$$S_1^T U + U S_1 = (\Pi_1 - \Sigma_1)^T (\Pi_1 - \Sigma_1) \tag{6.24}$$

$$S_1^T V + V S_1 = (\Pi_1 + \Sigma_1)^T (\Pi_1 + \Sigma_1). \tag{6.25}$$

Using Theorem 6.1, Corollary 6.1, and the duality argument in Section 6.1, it is not difficult to prove the following result:

Theorem 6.2 *The LMI (6.21) is feasible with $\nu > 0$ and $K = 0$ if and only if*

$$\sigma_r > 0, \quad \sigma_{r-1} < 0, \quad U - V > \frac{2}{\sigma_r}\Sigma_1^T \Sigma_1, \tag{6.26}$$

and

$$w^*(\Pi_2 - \Sigma_2)^T(\Pi_2 - \Sigma_2)w > w^*(\Pi_2 + \Sigma_2)^T(\Pi_2 + \Sigma_2)w \tag{6.27}$$

for every (possibly complex) eigenvector w of S_2.

If the restriction $K = 0$ is removed, then these conditions are relaxed as follows: When $r = 1$, the LMI (6.21) is feasible with $\nu > 0$ if and only if $U > V$ and (6.27) holds for every eigenvector w of S_2. When $r = 2$, $\sigma_r > 0$ and $U - V > \frac{2}{\sigma_r}\Sigma_1^T\Sigma_1$ are also required. When $r \geq 3$, all the conditions for feasibility with $K = 0$ above are required. □

Example 6.1 *With $y = x_1$, and the nondecreasing nonlinearity $\gamma(x_3) = x_3^3$, the system*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3^3 \\ \dot{x}_2 &= -x_2 + x_3 \\ \dot{x}_3 &= u, \end{aligned} \tag{6.28}$$

is of the form (6.20). Omitting $y = x_1$ and u from the right-hand side of (6.28), and using the change of variables $\xi_2 = -x_3$, $\xi_3 = x_2 - x_3$, we obtain

$$\begin{aligned} \dot{y} &= -\xi_2 + \xi_3 - \gamma(\xi_2) \\ \dot{\xi}_2 &= 0 \\ \dot{\xi}_3 &= -\xi_3, \end{aligned} \tag{6.29}$$

which is of the form (6.22). With the restriction $K = 0$, the observer design is not feasible because $\sigma_r = 0$. When the restriction $K = 0$ is removed, the design is feasible because $r = 1$, the eigenvalues of S are $\lambda_2 = 0$, $\lambda_3 = -1$, and (6.27) holds for the ξ_2-subsystem with $S_2 = 0$, $\Pi_2 = -1$, $\Sigma_2 = 1$. A solution of the LMI (6.21) is $L = [-0.6929 \ \ -0.5621 \ \ -0.4814]^T$, $K = -3.2417$ and, hence, the resulting observer is

$$\begin{aligned} \dot{\hat{x}}_1 &= -0.6929(\hat{x}_1 - y) + y + \hat{x}_2 + (\hat{x}_3 - 3.2417(\hat{x}_1 - y))^3 \\ \dot{\hat{x}}_2 &= -0.5621(\hat{x}_1 - y) - \hat{x}_2 + \hat{x}_3 \\ \dot{\hat{x}}_3 &= -0.4814(\hat{x}_1 - y) + u. \end{aligned} \tag{6.30}$$

Example 6.2 *Consider again the system (6.28), but with $-x_2$ in the second equation replaced with x_2:*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3^3 \\ \dot{x}_2 &= x_2 + x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{6.31}$$

With $y = x_1$, the omission of x_1 and u from the right-hand side, and the change of variables $\xi_1 = x_2 + x_3$, $\xi_2 = -x_3$, we get

$$\begin{aligned} \dot{y} &= \xi_1 + \xi_2 - \gamma(\xi_2) \\ \dot{\xi}_1 &= \xi_1 \\ \dot{\xi}_2 &= 0, \end{aligned} \tag{6.32}$$

where the eigenvalues of S are $\lambda_1 = 1$ and $\lambda_2 = 0$. The observer design is not feasible even with $K \neq 0$ because $U > V$ does not hold for the ξ_1-subsystem with $S_1 = 1$, $\Pi_1 = 1$, $\Sigma_1 = 0$. Thus, the sign in front of x_2 in the second equation of (6.28) is crucial. □

6.4 Feasibility of the Reduced-Order Observer

We now show that the feasibility conditions for the reduced-order observer design are the same as those for the full-order observer.

Theorem 6.3 *Let the constant $\nu \geq 0$ and the diagonal matrix $\Lambda > 0$ be given. Then, the following two statements are equivalent:*

1. *There exist matrices $P = P^T > 0$, K and L satisfying the full-order observer LMI (5.11).*
2. *There exist matrices $P_o = P_o^T > 0$ and N satisfying the reduced-order observer LMI (5.34).*

Proof:

(1 ⇒ 2) Suppose the full-order observer LMI (5.11) is feasible for the system (5.29). Partitioning P and L as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tag{6.33}$$

and substituting

$$A = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \qquad H = [\, H_1 \ H_2 \,], \quad \text{and} \quad C = [\, I \ \ 0 \,] \tag{6.34}$$

in (5.11), we obtain

$$(A + LC)^T P + P(A + LC) + \nu I = \tag{6.35}$$
$$\begin{bmatrix} \star & \star \\ \star & P_3(A_2 + P_3^{-1} P_2^T A_1) + (A_2 + P_3^{-1} P_2^T A_1)^T P_3 + \nu I \end{bmatrix} \leq 0$$

$$PG + (H + KC)^T \Lambda = \begin{bmatrix} \star \\ P_3(G_2 + P_3^{-1} P_2^T G_1) + H_2^T \Lambda \end{bmatrix} = 0. \tag{6.36}$$

Defining $P_o := P_3$, and $N := P_3^{-1} P_2^T$, we note that (6.35) and (6.36) imply

$$P_o(A_2 + NA_1) + (A_2 + NA_1)^T P_o + \nu I \leq 0 \tag{6.37}$$
$$P_o(G_2 + NG_1) + H_2^T \Lambda = 0, \tag{6.38}$$

which is the reduced-order observer LMI (5.34).

(2 ⇒ 1) Suppose (6.37) and (6.38) hold for the system (5.29). To prove the existence of $P = P^T > 0$, K and L satisfying (5.11), we rewrite system (5.29) in y and $\chi = x_o + Ny$ coordinates, so that

$$A + LC = \begin{bmatrix} \tilde{L}_1 & A_1 \\ \tilde{L}_2 & A_2 + NA_1 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 + NG_1 \end{bmatrix}, \tag{6.39}$$

$$H + KC = [\, H_1 - H_2 N + K \ \ H_2 \,],$$

where $\tilde{L}_1 := L_1 - A_1 N$ and $\tilde{L}_2 = L_2 - (A_2 + NA_1)N$. We let

$$P = \begin{bmatrix} I & 0 \\ 0 & P_o \end{bmatrix}, \tag{6.40}$$

and obtain

$$(A + LC)^T P + P(A + LC) = \begin{bmatrix} \tilde{L}_1 + \tilde{L}_1^T & A_1 + \tilde{L}_2^T P_o \\ P_o \tilde{L}_2 + A_1^T & P_o(A_2 + NA_1) + (A_2 + NA_1)^T P_o \end{bmatrix}$$
$$PG + (H + KC)^T \Lambda = \begin{bmatrix} G_1 + (H_1 - H_2 N + K)^T \Lambda \\ P_o(G_2 + NG_1) + H_2^T \Lambda \end{bmatrix}. \tag{6.41}$$

In view of (6.37) and (6.38), the choice

$$\tilde{L}_1 = -\frac{\nu}{2}I, \quad \tilde{L}_2 = -P_0^{-1}A_1^T, \quad K = -H_1 + H_2N - \Lambda^{-1}G_1^T \tag{6.42}$$

results in

$$\begin{aligned}&(A+LC)^TP + P(A+LC) + \nu I \leq 0\\ &PG + (H+KC)^T\Lambda = 0,\end{aligned}$$

which is the full-order observer LMI (5.11). □

6.5 Proof of Theorem 6.1

We prove the theorem in two steps. In Step 1 we prove that a state feedback control law $u = \mathcal{F}x$, rendering (6.11)-(6.13) SPR from w to z, exists if and only if $g_1 > 0$, $g_2 < 0$, and $Y = Y^T > \frac{2}{g_1}G_0G_0^T$ can be found such that (6.14) holds. In Step 2, we show that $Y = Y^T > \frac{2}{g_1}G_0G_0^T$ satisfying (6.14) exists if and only if $U - V > \frac{2}{g_1}G_0^1{G_0^1}^T$ and (6.19) holds for every eigenvector of ${A_0^2}^T$.

Step 1 - Necessity: Suppose $u = \mathcal{F}x$ renders (6.11)-(6.13) SPR from w to $z = y_1$, let $X := \mathcal{P}^{-1}$, and rewrite (6.9)-(6.10) as

$$(\mathcal{A}+\mathcal{B}\mathcal{F})X + X(\mathcal{A}+\mathcal{B}\mathcal{F})^T < 0 \tag{6.43}$$

$$X\mathcal{H}^T + \mathcal{G} = 0 \tag{6.44}$$

which, for system (6.11)-(6.13), yields

$$\left.\begin{array}{l}\mathcal{H} = [\,0_{1\times(n-r)} \;\; -1 \;\; 0 \;\; \cdots \;\; 0\,]\\ \mathcal{G}^T = [\,G_0 \;\; g_1 \;\; \cdots \;\; g_r\,]\end{array}\right\} \quad \Rightarrow \quad X = \begin{bmatrix} X_0 & G_0 & \star & \star \\ G_0^T & g_1 & \cdots & g_r \\ \star & \vdots & \star & \star \\ \star & g_r & \star & \star \end{bmatrix}. \tag{6.45}$$

Then, $g_1 > 0$ because $X > 0$, and $g_2 < 0$ because, from (6.11) and (6.45),

$$(\mathcal{A}+\mathcal{B}\mathcal{F})X + X(\mathcal{A}+\mathcal{B}\mathcal{F})^T = \begin{bmatrix} A_0X_0 + X_0A_0^T + E_0G_0^T + G_0E_0^T & \star & \star & \star \\ \star & 2g_2 & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix} < 0. \tag{6.46}$$

This also shows that $Y = 2X_0$ satisfies (6.14). Then, $Y > \frac{2}{g_1} G_0 G_0^T$ follows from (6.45) because

$$\begin{bmatrix} X_0 & G_0 \\ G_0^T & g_1 \end{bmatrix} > 0, \tag{6.47}$$

and the Schur complement of g_1 is $X_0 - \frac{1}{g_1} G_0 G_0^T > 0$.

Step 1 - Sufficiency: For $r = 1$, the equations (6.11)-(6.13) are

$$\begin{aligned} \dot{\xi} &= A_0 \xi + E_0 y_1 + G_0 w & (6.48) \\ \dot{y}_1 &= u + g_1 w. & (6.49) \end{aligned}$$

For this system $\mathcal{H} = [0_{1\times(n-r)} \; -1]$, and (6.44) is satisfied by

$$X = \begin{bmatrix} \frac{1}{2} Y & G_0 \\ G_0^T & g_1 \end{bmatrix} > 0 \tag{6.50}$$

where Y is as in (6.14). Substituting

$$\mathcal{F} = \left[\kappa G_0^T Y^{-1} \;\; -\frac{\kappa}{2} \right], \quad \kappa > 0 \tag{6.51}$$

in (6.43), we obtain

$$(\mathcal{A} + \mathcal{B}\mathcal{F})X + X(\mathcal{A} + \mathcal{B}\mathcal{F})^T = \tag{6.52}$$
$$\begin{bmatrix} \frac{1}{2} A_0 Y + \frac{1}{2} Y A_0^T + E_0 G_0^T + G_0 E_0^T & A_0 G_0 + g_1 E_0 \\ (A_0 G_0 + g_1 E_0)^T & -\kappa(g_1 - 2\, G_0^T Y^{-1} G_0) \end{bmatrix}.$$

From (6.50), $g_1 - 2 G_0^T Y^{-1} G_0 > 0$ and, hence, the right-hand side of (6.52) is rendered negative definite by selecting $\kappa > 0$ sufficiently large.

For $r \geq 2$, we need the following lemma proved at the end of the section:

Lemma 6.1 *With input $w \in I\!R$ and output $z \in I\!R$, the system*

$$\begin{aligned} \dot{\eta} &= M\eta + Nz & (6.53) \\ \dot{z} &= L\eta - az + gw & (6.54) \end{aligned}$$

is SPR if and only if $a > 0$, $g > 0$, and the $\mathcal{L}_2$-gain of the system

$$\begin{aligned} \dot{\eta} &= \left(M + \frac{1}{2a} NL \right) \eta + Nv & (6.55) \\ y &= L\eta \end{aligned}$$

from input v to output y is $\gamma < 2a$. □

With the change of coordinates

$$z = y_1, \quad \bar{\xi} = \xi - \frac{1}{g_1} G_0 y_1, \quad \bar{y}_i = y_i - \frac{g_i}{g_1} y_1, \quad i = 2, \cdots, r, \tag{6.56}$$

the system (6.11)-(6.13) is rewritten as

$$\begin{aligned}
\dot{\bar{\xi}} &= A_0 \bar{\xi} - \frac{1}{g_1} G_0 \bar{y}_2 + \left(\frac{1}{g_1} A_0 G_0 + E_0 - \frac{g_2}{g_1^2} G_0 \right) z && (6.57)\\
\dot{\bar{y}}_i &= \bar{y}_{i+1} - \frac{g_i}{g_1} \bar{y}_2 - \frac{g_i g_2}{g_1^2} z, \quad i = 2, \cdots, r-1, && (6.58)\\
\dot{\bar{y}}_r &= u - \frac{g_r}{g_1} \bar{y}_2 - \frac{g_r g_2}{g_1^2} z && (6.59)\\
\dot{z} &= \bar{y}_2 + \frac{g_2}{g_1} z + g_1 w, && (6.60)
\end{aligned}$$

which is (6.53)-(6.54) with $\eta = (\bar{\xi}^T, \bar{y}_2, \cdots, \bar{y}_r)^T$, $L\eta = \bar{y}_2$, $a = -\frac{g_2}{g_1}$, and $g = g_1$. Since $g_1 > 0$ and $g_2 < 0$, we conclude that $a > 0$ and $g > 0$ as in Lemma 6.1. The system (6.55) in Lemma 6.1 is

$$\begin{aligned}
\dot{\bar{\xi}} &= A_0 \bar{\xi} + R_0 \bar{y}_2 + Q_0 v && (6.61)\\
\dot{\bar{y}}_i &= \bar{y}_{i+1} - \frac{g_i}{2g_1} \bar{y}_2 - \frac{g_i g_2}{g_1^2} v, \quad i = 2, \cdots, r-1, && (6.62)\\
\dot{\bar{y}}_r &= u - \frac{g_r}{2g_1} \bar{y}_2 - \frac{g_r g_2}{g_1^2} v && (6.63)\\
y &= \bar{y}_2, && (6.64)
\end{aligned}$$

where

$$R_0 = -\left(\frac{1}{2g_2} A_0 G_0 + \frac{g_1}{2g_2} E_0 + \frac{1}{2g_1} G_0 \right) \quad \text{and} \quad Q_0 = \left(\frac{1}{g_1} A_0 G_0 + E_0 - \frac{g_2}{g_1^2} G_0 \right). \tag{6.65}$$

Next, we show that there exists a state feedback control law for u that assigns an $\mathcal{L}_2$-gain $\gamma < 2a$ from input v to output $y = \bar{y}_2$. It was shown by Isidori [27, Section 13.2] and Chen [10, Chapter 5] that a control law ensuring $\gamma < 2a$ exists if and only if there exists a matrix $Z = Z^T > 0$ satisfying

$$A_0 Z + Z A_0^T + \frac{1}{(2a)^2} Q_0 Q_0^T - R_0 R_0^T < 0. \tag{6.66}$$

Substituting $a = -\frac{g_2}{g_1}$, Q_0 and R_0 from (6.65), it is not difficult to verify that (6.66) holds for

$$Z = -\frac{1}{4g_2} \left(Y - \frac{2}{g_1} G_0 G_0^T \right), \tag{6.67}$$

where Y satisfies (6.14). Moreover, $Z > 0$ because $g_2 < 0$ and $Y > \frac{2}{g_1}G_0G_0^T$. This means that there exists $u = \mathcal{F}x$ such that $\gamma < 2a$, thus rendering (6.11)-(6.13) SPR from w to z, by Lemma 6.1.

Step 2 - Necessity: Suppose that

$$Y = Y^T = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} > \frac{2}{g_1}G_0G_0^T \tag{6.68}$$

satisfies (6.14). Then, $Y_{ii} > \frac{2}{g_1}G_0^i{G_0^i}^T$, $i = 1, 2, 3$, and

$$A_0^iY_{ii} + Y_{ii}{A_0^i}^T < -(E_0^i + G_0^i)(E_0^i + G_0^i)^T + (E_0^i - G_0^i)(E_0^i - G_0^i)^T. \tag{6.69}$$

Because $-A_0^1$ is Hurwitz, it follows that there exists $\tilde{Y}_{11} > Y_{11} > \frac{2}{g_1}G_0^1{G_0^1}^T$ satisfying

$$A_0^1\tilde{Y}_{11} + \tilde{Y}_{11}{A_0^1}^T = -(E_0^1 + G_0^1)(E_0^1 + G_0^1)^T + (E_0^1 - G_0^1)(E_0^1 - G_0^1)^T. \tag{6.70}$$

From (6.16) and (6.17), the solution of (6.70) is $\tilde{Y}_{11} = U - V$, and $U - V > \frac{2}{g_1}G_0^1{G_0^1}^T$ follows from $\tilde{Y}_{11} > \frac{2}{g_1}G_0^1{G_0^1}^T$.

For (6.69) with $i = 2$, the following lemma due to Scherer [82, Theorem 4] shows that (6.19) holds for every eigenvector w of ${A_0^2}^T$:

Lemma 6.2 *Let all the eigenvalues of A be on the imaginary axis, and let Q be an arbitrary matrix. Then, there exists $X = X^T > 0$ such that*

$$AX + XA^T < Q \tag{6.71}$$

if and only if

$$w^*Qw > 0 \tag{6.72}$$

for every eigenvector w of A^T. Moreover, if (6.72) holds, then for any X_0, there exists $X > X_0$ satisfying (6.71). □

Step 2 - Sufficiency: Let $Y_{33} = Y_{33}^T > 0$ be the solution of $A_0^3Y_{33} + Y_{33}{A_0^3}^T = -kI$, and denote

$$\tilde{A}_0 = \begin{bmatrix} A_0^1 & 0 \\ 0 & A_0^2 \end{bmatrix}, \quad \tilde{E}_0 = \begin{bmatrix} E_0^1 \\ E_0^2 \end{bmatrix}, \quad \tilde{G}_0 = \begin{bmatrix} G_0^1 \\ G_0^2 \end{bmatrix}. \tag{6.73}$$

If there exists $\tilde{Y} = \tilde{Y}^T > \frac{2}{g_1}\tilde{G}_0\tilde{G}_0^T$ such that

$$\tilde{A}_0\tilde{Y} + \tilde{Y}\tilde{A}_0^T < -(\tilde{E}_0 + \tilde{G}_0)(\tilde{E}_0 + \tilde{G}_0)^T + (\tilde{E}_0 - \tilde{G}_0)(\tilde{E}_0 - \tilde{G}_0)^T, \tag{6.74}$$

then

$$Y = \begin{bmatrix} \tilde{Y} & 0 \\ 0 & Y_{33} \end{bmatrix} \tag{6.75}$$

satisfies (6.14), and $Y > \frac{2}{g_1} G_0 G_0^T$ for large enough $k > 0$. To prove (6.74), we use

$$\begin{aligned} A_0^1 Y_{11} + Y_{11} A_0^{1T} &< -(E_0^1 + G_0^1)(E_0^1 + G_0^1)^T + (E_0^1 - G_0^1)(E_0^1 - G_0^1)^T & (6.76) \\ A_0^2 Y_{22} + Y_{22} A_0^{2T} &< -(E_0^2 + G_0^2)(E_0^2 + G_0^2)^T + (E_0^2 - G_0^2)(E_0^2 - G_0^2)^T, & (6.77) \end{aligned}$$

where $\frac{2}{g_1} G_0^1 G_0^{1T} < Y_{11} < \tilde{Y}_{11} = U - V$ because $-A_0^1$ is Hurwitz, and $Y_{22} = Y_{22}^T > 0$ in view of Lemma 6.2 and (6.19). Then

$$\tilde{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \tag{6.78}$$

satisfies

$$\begin{aligned} \tilde{A}_0\tilde{Y} + \tilde{Y}\tilde{A}_0 + (\tilde{E}_0 + \tilde{G}_0)(\tilde{E}_0 + \tilde{G}_0)^T - (\tilde{E}_0 - \tilde{G}_0)(\tilde{E}_0 - \tilde{G}_0)^T &= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \\ Q_1 = A_0^1 Y_{11} + Y_{11} A_0^{1T} + (E_0^1 + G_0^1)(E_0^1 + G_0^1)^T - (E_0^1 - G_0^1)(E_0^1 - G_0^1)^T &< 0 \\ Q_2 = A_0^2 Y_{22} + Y_{22} A_0^{2T} + (E_0^2 + G_0^2)(E_0^2 + G_0^2)^T - (E_0^2 - G_0^2)(E_0^2 - G_0^2)^T &< 0, \end{aligned}$$

and Y_{12} satisfies the corresponding Sylvester equation. Finally, $\tilde{Y} > \frac{2}{g_1} \tilde{G}_0 \tilde{G}_0^T$ follows because $Y_{11} > \frac{2}{g_1} G_0^1 G_0^{1T}$ and, from Lemma 6.2, Y_{22} can be selected sufficiently large.
□

Proof of Lemma 6.1

Let the system (6.53)-(6.54) be SPR, that is, let $P = P^T > 0$ exist such that

$$\begin{aligned} A^T P + PA &< 0 & (6.79) \\ PB &= C^T, & (6.80) \end{aligned}$$

where

$$A = \begin{bmatrix} M & N \\ L & -a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T. \tag{6.81}$$

Because of (6.80), P has the form

$$P = \frac{1}{g} \begin{bmatrix} P_0 & 0 \\ 0 & 1 \end{bmatrix} > 0, \tag{6.82}$$

which implies $g > 0$. Using (6.81) and (6.82),

$$A^T P + PA = \begin{bmatrix} P_0 M + M^T P_0 & P_0 N + L^T \\ N^T P_0 + L & -2a \end{bmatrix} < 0, \tag{6.83}$$

from which $a > 0$, and the Schur complement of $-2a$ is

$$P_0 \left(M + \frac{1}{2a} NL \right) + \left(M + \frac{1}{2a} NL \right)^T P_0 + \frac{1}{2a} P_0 N N^T P_0 + \frac{1}{2a} L^T L < 0. \tag{6.84}$$

From the *bounded real lemma*, (6.84) holds if and only if the $\mathcal{L}_2$-gain of the system (6.55) is $\gamma < 2a$.

To prove the converse, we note that if the $\mathcal{L}_2$-gain of the system (6.55) is $\gamma < 2a$, then there exists $P_0 = P_0^T > 0$ satisfying (6.84). Because $a > 0$ and $g > 0$, (6.82) satisfies (6.79) and (6.80). □

6.6 Summary

We have derived necessary and sufficient conditions for the feasibility of the observer design. The result is established by showing that the observer design is dual to a control design in which a linear system is rendered SPR with respect to a disturbance input. We have proved that the feasibility of the reduced-order observer design coincides with that of the full-order observer.

Chapter 7

Robust Output-Feedback Design

As with any other observer, the main purpose of the new observer is stabilization and tracking in conjunction with a control law $u = \alpha(y, \hat{x}, t)$ which employs the available measurement y, the state estimate $\hat{x}$ and, possibly, exogenous signals. Because the separation principle does not hold for nonlinear systems, our observer is to be used with control laws that guarantee boundedness of the states for bounded observer errors. This design strategy is detailed in Section 7.1, and illustrated on an analytical example. In Section 7.2 we study the effect of unmodeled dynamics, and propose a small-gain design for robust output-feedback control. This small-gain design is illustrated on the jet engine compressor example in Section 7.3.

7.1 Observer-Based Control Design

We view the state observer error $e = x - \hat{x}$ as a disturbance acting on the system (5.1) through the control law $u = \alpha(y, x - e, t)$ and require that the control law guarantee one of the following properties:

No Finite Escape (NFE) Property:

$$e(t) \in \mathcal{L}_\infty \quad \Rightarrow \quad x(t) \in \mathcal{L}_\infty^e \,, \quad \forall x(0) \in I\!R^n. \tag{7.1}$$

Bounded Error - Bounded State (BEBS) Property:

$$e(t) \in \mathcal{L}_\infty \quad \Rightarrow \quad x(t) \in \mathcal{L}_\infty \,, \quad \forall x(0) \in I\!R^n. \tag{7.2}$$

If $u = \alpha(y, \hat{x}, t)$ is a NFE control law for the system (5.1), then the assumption that $x(t) \in \mathcal{L}_\infty^e$ made in Theorem 5.1 holds. To see this, suppose that the maximal interval of existence is finite. On this interval, $e(t)$ is bounded because of (5.16). If $x(t)$ were unbounded, this would contradict the NFE property (7.1), thus $x(t)$ exists for all

$t \geq 0$. Likewise, if $u = \alpha(y, \hat{x}, t)$ is a BEBS control law, then $x(t) \in \mathcal{L}_\infty$, as required in Theorem 5.2. To guarantee $x(t) \in \mathcal{L}_\infty^e$ for the system (5.38), the NFE property is to hold for all bounded disturbances $\mu(t)$.

The BEBS property incorporates the NFE property. Requiring these properties is meaningful because BEBS control laws have already been designed for classes of nonlinear systems by Freeman and Kokotović [18], Krstić *et al.* [53], and Marino and Tomei [63]. We now illustrate the use of our observer for output-feedback design in conjunction with the *observer backstepping* design of Krstić *et al.* [53].

Example 7.1 *For the system*

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 \\ \dot{x}_2 &= x_2 + x_3 - \exp(x_2) + u \\ \dot{x}_3 &= 2u\,, \end{aligned} \tag{7.3}$$

studied in Example 5.1, suppose that the output $y = x_1$ is required to track $y_d(t)$, where $y_d(t)$, $\dot{y}_d(t)$, and $\ddot{y}_d(t)$ are known, continuous and bounded on $[0, \infty)$. Using the observer (5.28), we design a BEBS control law $u = \alpha(y, \hat{x}, t)$ that ensures asymptotic tracking and boundedness of the states. Since the system (7.3) with output y has relative degree two, we apply two steps of observer backstepping as in Krstić et al. *[53, Section 7.1].*

Step 1. *We let $\zeta_1 := y - y_d$, substitute $x_2 = \hat{x}_2 + e_2$ and $x_1 = y$ in the first equation of (7.3), and obtain*

$$\dot{\zeta}_1 = \hat{x}_2 + y^2 - \dot{y}_d + e_2\,. \tag{7.4}$$

For $\hat{x}_2$, we design the 'virtual' control law

$$\alpha_0(y, y_d, \dot{y}_d) = -c_1\zeta_1 - y^2 + \dot{y}_d\,, \quad c_1 > 0, \tag{7.5}$$

which results in

$$\dot{\zeta}_1 = -c_1\zeta_1 + \zeta_2 + e_2\,, \tag{7.6}$$

where $\zeta_2 := \hat{x}_2 - \alpha_0(y, y_d, \dot{y}_d)$.

Step 2. *Using the observer equation (5.28), we get*

$$\dot{\zeta}_2 = u + g(y, y_d, \dot{y}_d, \ddot{y}_d, \hat{x}) - \frac{\partial \alpha_0}{\partial y} e_2\,, \tag{7.7}$$

where

$$
\begin{aligned}
g(y, y_d, \dot{y}_d, \ddot{y}_d, \hat{x}) &= \hat{x}_2 + \hat{x}_3 - \exp(\hat{x}_2 - (\hat{x}_1 - y)) - 4(\hat{x}_1 - y) \\
&\quad - \frac{\partial \alpha_0}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_0}{\partial \dot{y}_d}\ddot{y}_d - \frac{\partial \alpha_0}{\partial y}\left(\hat{x}_2 + y^2\right). \qquad (7.8)
\end{aligned}
$$

We let $c_2 > 0$, $d > 0$, *and*

$$
u = \alpha(y, \hat{x}, t) = -\zeta_1 - c_2\zeta_2 - d\left(\frac{\partial \alpha_0}{\partial y}\right)^2 \zeta_2 - g(y, y_d, \dot{y}_d, \ddot{y}_d, \hat{x}), \qquad (7.9)
$$

which yields

$$
\dot{\zeta}_2 = -\zeta_1 - c_2\zeta_2 - d\left(\frac{\partial \alpha_0}{\partial y}\right)^2 \zeta_2 - \frac{\partial \alpha_0}{\partial y} e_2 . \qquad (7.10)
$$

We prove that (7.9) is a BEBS control law with the Lyapunov function $W(\zeta_1, \zeta_2) = \frac{1}{2}\zeta_1^2 + \frac{1}{2}\zeta_2^2$. *Differentiation along (7.6), (7.10), and completion of squares yield*

$$
\dot{W} \leq -\frac{c_1}{2}\zeta_1^2 - c_2\zeta_2^2 + \left(\frac{1}{2c_1} + \frac{1}{4d}\right) e_2^2 \leq -c_0 W + d_0\, e_2^2 , \qquad (7.11)
$$

where $c_0 = \min\{c_1, 2c_2\}$ *and* $d_0 = \frac{1}{4d} + \frac{1}{2c_1}$. *From (7.11), boundedness of* $e(t)$ *guarantees that* ζ_1, ζ_2 *and, hence,* x_1 *and* x_2 *are bounded. To see that* x_3 *is also bounded, we define the new variable* $\eta := x_3 - 2x_2$, *governed by*

$$
\dot{\eta} = -2\eta - 6x_2 + 2\exp(x_2) . \qquad (7.12)
$$

Since x_2 *is bounded, (7.12) guarantees boundedness of* η, *therefore* x_3 *is also bounded.*

The conditions of Theorem 5.2 being satisfied, the observer (5.28) guarantees $e(t) \to 0$ *as* $t \to \infty$. *From (7.11), we see that* $e(t) \to 0$ *ensures* $\zeta(t) \to 0$ *as* $t \to \infty$. *Since* $\zeta_1 = y - y_d$, *we conclude that the observer-based control law (7.9) guarantees* $y(t) \to y_d(t)$ *as* $t \to \infty$. $\square$

7.2 Robust Output-Feedback Stabilization

We now consider the problem of output-feedback stabilization for the locally Lipschitz system

$$
\begin{aligned}
\dot{x} &= Ax + G[\gamma(Hx) + \Delta(Hx)\mu] + \varrho(y, u) , \quad y = Cx , &(7.13)\\
\dot{\xi} &= q(\xi, h(x)) &(7.14)\\
\mu &= p(\xi, h(x)) ,
\end{aligned}
$$

where $\gamma(\cdot)$ and $\Delta(\cdot)$ are as in (5.40), and the ξ-subsystem (7.14) represents unmodeled dynamics. This formulation extends the applicability of our observer because, if there is a subsystem to which the observer is not applicable, it can be treated as unmodeled dynamics.

The unmodeled dynamics (7.14) are assumed to possess the following input-to-output stability (IOS), and ISS properties:

$$|\mu(t)| \leq \max\left\{\beta_\mu(|\xi(0)|, t), \rho_{\mu h}\left(\sup_{0\leq\tau\leq t} |h(x(\tau))|\right)\right\} \tag{7.15}$$

$$|\xi(t)| \leq \max\left\{\beta_\xi(|\xi(0)|, t), \rho_{\xi h}\left(\sup_{0\leq\tau\leq t} |h(x(\tau))|\right)\right\}, \tag{7.16}$$

where $\beta_\mu(\cdot,\cdot)$, $\beta_\xi(\cdot,\cdot)$ are class-$\mathcal{KL}$ functions, and $\rho_{\mu h}(\cdot)$, $\rho_{\xi h}(\cdot)$ are class-$\mathcal{K}$ functions.

The ISS property of the observer error (5.41) is now rewritten in the 'max' form of Teel [93], which is suitable for the small-gain analysis we will employ. Using the fact that for each $\rho(\cdot) \in \mathcal{K}_\infty$, $\forall a, b \geq 0$, $a + b \leq \max\{(I + \rho)(a), (I + \rho^{-1})(b)\}$, where $I(\cdot) = (\cdot)$ represents the identity function, we get

$$|e(t)| \leq \max\left\{\beta_e(|e(0)|, t), \rho_{e\mu}\left(\sup_{0\leq\tau\leq t} |\mu(\tau)|\right)\right\}. \tag{7.17}$$

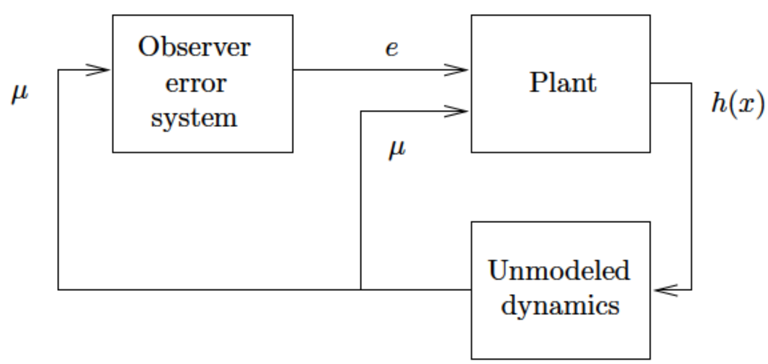

Figure 7.1: Closed-loop system with observer feedback.

To prepare for a small-gain design of $u = \alpha(y, \hat{x})$, we represent the closed-loop system with the block-diagram in Figure 7.1. The gains of the unmodeled dynamics block and the observer error block are $\rho_{\mu h}(\cdot)$ and $\rho_{e\mu}(\cdot)$, respectively. Let $\rho_{he}(\cdot)$ and $\rho_{h\mu}(\cdot)$ be the plant gains from e and μ to $h(x)$, respectively. The task for the control law $u = \alpha(y, \hat{x})$ is to render $\rho_{he}(\cdot)$ and $\rho_{h\mu}(\cdot)$ small enough for the inner loop gain $\rho_{h\mu} \circ \rho_{\mu h}(\cdot)$, and the outer loop gain $\rho_{he} \circ \rho_{e\mu} \circ \rho_{\mu h}(\cdot)$ to satisfy, for all $s > 0$,

$$\rho_{h\mu} \circ \rho_{\mu h}(s) < s \tag{7.18}$$

$$\rho_{he} \circ \rho_{e\mu} \circ \rho_{\mu h}(s) < s\,. \tag{7.19}$$

Then, GAS of the closed-loop system will be guaranteed as in the *nonlinear small-gain theorem* of Teel *et al.* [39, 93].

Theorem 7.1 *Consider the system (7.13)-(7.14), in which $\gamma(\cdot)$ and $\Delta(\cdot)$ satisfy (5.40), and the ξ-subsystem satisfies (7.15) and (7.16). Suppose that the observer*

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + G\,\gamma(H\hat{x} + K(C\hat{x} - y)) + \varrho(y, u) \tag{7.20}$$

is such that the LMI (5.11) holds with a matrix $P = P^T > 0$, a constant $\nu > 0$, and a diagonal matrix $\Lambda > 0$. If the control law $u = \alpha(y, \hat{x})$ guarantees

$$|h(x(t))| \;\leq\; \max\left\{\beta_h(|x(0)|, t),\; \rho_{h\mu}\left(\sup_{0\leq\tau\leq t}|\mu(\tau)|\right),\; \rho_{he}\left(\sup_{0\leq\tau\leq t}|e(\tau)|\right)\right\} \tag{7.21}$$

$$|x(t)| \;\leq\; \max\left\{\beta_x(|x(0)|, t),\; \rho_{x\mu}\left(\sup_{0\leq\tau\leq t}|\mu(\tau)|\right),\; \rho_{xe}\left(\sup_{0\leq\tau\leq t}|e(\tau)|\right)\right\} \tag{7.22}$$

where $\rho_{h\mu}(\cdot)$ and $\rho_{he}(\cdot)$ satisfy (7.18) and (7.19), respectively, then the origin of the closed-loop system (7.13), (7.14), (7.20) is globally asymptotically stable.

Proof: Using the notation $\|h\|_t := \sup_{0\leq\tau\leq t}|h(x(\tau))|$ for each t in the maximal interval of existence $[0, t_f)$, and substituting (7.15) in (7.17), we get

$$\|e\|_t \leq \max\left\{\beta_e(|e(0)|, 0),\; \rho_{e\mu} \circ \beta_\mu(|\xi(0)|, 0),\; \rho_{e\mu} \circ \rho_{\mu h}(\|h\|_t)\right\}. \tag{7.23}$$

We then substitute (7.23) and (7.17) in (7.21), and obtain

$$\|h\|_t \leq \max\left\{\delta_h(|x(0)|, |\xi(0)|, |e(0)|),\; \rho_{h\mu} \circ \rho_{\mu h}(\|h\|_t),\; \rho_{he} \circ \rho_{e\mu} \circ \rho_{\mu h}(\|h\|_t)\right\}, \tag{7.24}$$

where

$$\delta_h(|x(0)|, |\xi(0)|, |e(0)|) := \tag{7.25}$$
$$\max\left\{\beta_h(|x(0)|, 0),\; \rho_{h\mu} \circ \beta_\mu(|\xi(0)|, 0),\; \rho_{he} \circ \beta_e(|e(0)|, 0),\; \rho_{he} \circ \rho_{e\mu} \circ \beta_\mu(|\xi(0)|, 0)\right\}.$$

The substitution of the small-gain conditions (7.18) and (7.19) in (7.24) results in

$$\|h\|_t \leq \delta_h(|x(0)|, |\xi(0)|, |e(0)|)\,, \tag{7.26}$$

and, from (7.15) and (7.23), it is not difficult to derive functions δ_μ and δ_e of the initial conditions such that $\|\mu\|_t \leq \delta_\mu$ and $\|e\|_t \leq \delta_e$. Then, in view of (7.16) and (7.22), we can find a class-$\mathcal{K}$ function $\Omega(\cdot)$ such that, for each $t \in [0, t_f)$,

$$|(x(t), \xi(t), e(t))| \leq \Omega(|(x(0), \xi(0), e(0))|)\,. \tag{7.27}$$

Since the right hand side of (7.27) is independent of t, we conclude that $t_f = \infty$, and the origin is Lyapunov stable.

To prove GAS, the remaining task is to prove convergence of the solutions to the origin. To this end, we denote $\|h\|_a := \limsup_{t\to\infty} |h(x(t))|$ as in Teel [93], and note from (7.27) that $\|h\|_a$, $\|\mu\|_a$ and $\|e\|_a$ are finite. Then, from (7.15), (7.23) and (7.21),

$$\begin{aligned}
\|\mu\|_a &\leq \rho_{\mu h}(\|h\|_a) && (7.28)\\
\|e\|_a &\leq \rho_{e\mu} \circ \rho_{\mu h}(\|h\|_a) && (7.29)\\
\|h\|_a &\leq \max\{\rho_{h\mu}(\|\mu\|_a), \rho_{he}(\|e\|_a)\}\,. && (7.30)
\end{aligned}$$

The substitution of (7.28) and (7.29) in (7.30), and the use of (7.18) and (7.19) yield $\|\mu\|_a = \|e\|_a = \|h\|_a = 0$, thus proving the convergence of $\mu(t)$, $e(t)$ and $h(x(t))$ to zero. Then, the ISS conditions (7.16) and (7.22) imply $(x(t), \xi(t), e(t)) \to 0$. $\square$

7.3 Design Example

An axial compressor model, which has been the starting point for jet engine control studies, is the following single-mode approximation of a PDE model due to Moore and Greitzer [65],

$$\begin{aligned}
\dot{\phi} &= -\psi + \frac{3}{2}\phi + \frac{1}{2} - \frac{1}{2}(\phi+1)^3 - 3(\phi+1)R && (7.31)\\
\dot{\psi} &= \frac{1}{\beta^2}(\phi + 1 - u) && (7.32)\\
\dot{R} &= \sigma R(-2\phi - \phi^2 - R)\,, \quad R(0) \geq 0\,, && (7.33)
\end{aligned}$$

where ϕ and ψ are the deviations of the mass flow and the pressure rise from their set points, the control input u is the flow through the throttle, and, σ and β are positive constants. This model captures the main surge instability between the mass flow and the pressure rise. It also incorporates the nonnegative magnitude R of the first stall mode.

Krstić *et al.* designed a state feedback GAS control law in [53, Section 2.4], and later replaced it by a design using ϕ and ψ in [52]. With a high-gain observer, Isidori

[27, Section 12.7], and Maggiore and Passino [58], obtained a semiglobal result using the measurement of ψ alone. With $y = \psi$, we will now achieve GAS for (7.31)-(7.33). The exact observer cannot be designed because of the nonlinearities ϕR and $\phi^2 R$. However, the (ϕ, ψ)-subsystem (7.31),(7.32) contains the nondecreasing nonlinearity $(\phi + 1)^3$, and is of the form (7.13) with disturbance $\mu = R$. This suggests that we treat the R-subsystem (7.33) as unmodeled dynamics and apply the design of Section 7.2.

First, we prove that $\mu = R$ satisfies the IOS property (7.15) with $h(x) = \phi$ as the input. With $V = R^2$ as an ISS-Lyapunov function, $R \geq 2.1|\phi|$ implies $\dot{V} \leq -0.09\sigma R^3$, because $R(t) \geq 0$ for all $t \geq 0$. This means that (7.15) holds with the linear gain

$$\rho_{\mu h}(\cdot) = 2.1(\cdot)\,, \tag{7.34}$$

and, since $\mu = \xi = R$, the ISS property (7.16) is also satisfied.

To design the reduced-order observer of Section 5.2 for the (ϕ, ψ)-subsystem, we let $\chi = \phi + N\psi$, and obtain

$$\dot{\chi} = \left(\frac{3}{2} + \frac{N}{\beta^2}\right)\chi - \frac{1}{2}(\chi - N\psi + 1)^3 - 3(\chi - N\psi + 1)R + \bar{\varrho}(\psi, u)\,, \tag{7.35}$$

where

$$\bar{\varrho}(\psi, u) := -\left(\frac{3}{2} + \frac{N}{\beta^2}\right)N\psi - \psi + \frac{1}{2} + \frac{N}{\beta^2}(1 - u)\,. \tag{7.36}$$

The resulting observer is the scalar equation

$$\begin{aligned}
\dot{\hat{\chi}} &= \left(\frac{3}{2} + \frac{N}{\beta^2}\right)\hat{\chi} - \frac{1}{2}(\hat{\chi} - N\psi + 1)^3 + \bar{\varrho}(\psi, u) \\
\hat{\phi} &= \hat{\chi} - N\psi\,.
\end{aligned} \tag{7.37}$$

For its implementation, the LMI (5.34) is satisfied by selecting N such that

$$k := -\left(\frac{3}{2} + \frac{N}{\beta^2}\right) > 0\,. \tag{7.38}$$

To prove the ISS property (7.17) for the observer error $e_\phi = \phi - \hat{\phi}$, we employ the ISS-Lyapunov function $V_e = e_\phi^2$, and evaluate its derivative for

$$\dot{e}_\phi = -ke_\phi - \frac{1}{2}(a^3 - b^3 + 6aR)\,, \tag{7.39}$$

where $a := \chi - N\psi + 1$ and $b = \hat{\chi} - N\psi + 1$. Employing the inequality (5.47), and substituting $a - b = e_\phi$, we get

$$\dot{V}_e \leq -2ke_\phi^2 + 12R^2\,, \tag{7.40}$$

from which $|e_\phi| \geq \sqrt{\frac{6.1}{k}}|R|$ implies $\dot{V}_e \leq -0.03ke_\phi^2$ and, hence, the ISS property (7.17) holds with the linear gain

$$\rho_{e\mu}(\cdot) = \sqrt{\frac{6.1}{k}}(\cdot). \tag{7.41}$$

Clearly, the gain $\rho_{e\mu}(\cdot)$ can be rendered as small as desired by making k sufficiently large with a choice of N in (7.38).

We are now ready to design a control law as in Theorem 7.1. Noting that (7.31)-(7.32) is in strict feedback form, we apply one step of observer backstepping. For ψ, we design the virtual control law $\alpha_0 = c_1\hat{\phi}$. Denoting

$$\omega := \psi - c_1\hat{\phi} = \psi - c_1\phi + c_1 e_\phi, \tag{7.42}$$

we rewrite (7.31) as

$$\dot{\phi} = -c_1\phi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3\phi R - \omega - 3R + c_1 e_\phi\,. \tag{7.43}$$

The substitution of (7.37) in (7.42) yields $\omega = (1 + Nc_1)\psi - c_1\hat{\chi}$, and, from (7.32) and (7.37),

$$\dot{\omega} = \frac{1 + Nc_1}{\beta^2}\phi + \frac{1}{\beta^2}(1-u) + \Gamma(\hat{\phi}, \psi)\,, \tag{7.44}$$

where $\Gamma(\hat{\phi}, \psi) := c_1\psi + c_1 k\hat{\phi} + \frac{c_1}{2}(\hat{\phi}+1)^3 - \frac{c_1}{2}$. Then, the control law

$$u = 1 + (1 + Nc_1)\hat{\phi} + \beta^2(c_2\omega + \Gamma(\hat{\phi}, \psi)) \tag{7.45}$$

is implementable using the signals ψ and $\hat{\phi}$, and results in

$$\dot{\omega} = -c_2\omega + \frac{1 + Nc_1}{\beta^2}e_\phi. \tag{7.46}$$

The remaining task is to select the design parameters c_1 and c_2 such that (7.21) and (7.22) are satisfied. For the ISS-Lyapunov function $W(\phi, \omega) := \frac{1}{2}\phi^2 + \frac{1}{2}\omega^2$, the inequalities $-\frac{3}{2}\phi^3 \leq \frac{9}{8}\phi^2 + \frac{1}{2}\phi^4$, $-\phi\omega \leq \frac{1}{2}\phi^2 + \frac{1}{2}\omega^2$, $-3\phi R \leq \frac{9}{4}\phi^2 + R^2$, $-3\phi^2 R \leq 0$ (because $R(t) \geq 0$), $c_1\phi e_\phi \leq \frac{c_1}{2}\phi^2 + \frac{c_1}{2}e_\phi^2$, and $\frac{1+Nc_1}{\beta^2}\omega e_\phi \leq \frac{(1+Nc_1)^2}{2\beta^4 c_1}\omega^2 + \frac{c_1}{2}e_\phi^2$, yield

$$\dot{W} \leq -\left(\frac{c_1}{2} - \frac{31}{8}\right)\phi^2 - \left(c_2 - \frac{1}{2} - \frac{(1+Nc_1)^2}{2\beta^4 c_1}\right)\omega^2 + R^2 + c_1 e_\phi^2\,. \tag{7.47}$$

We let $c > 0$, and select c_1 and c_2 to satisfy

$$\left(\frac{c_1}{2} - \frac{31}{8}\right) > c, \quad \left(c_2 - \frac{1}{2} - \frac{(1+Nc_1)^2}{2\beta^4 c_1}\right) > c, \tag{7.48}$$

so that

$$\dot{W} \leq -c(\phi^2 + \omega^2) + R^2 + (2c + \frac{31}{4})e_\phi^2\,, \tag{7.49}$$

from which (7.22) follows for $x = (\phi, \psi)$. For $h(x) = \phi$ and $\mu = R$, we now compute the gains $\rho_{h\mu}(\cdot)$ and $\rho_{he}(\cdot)$ in (7.21). Using the fact that for each constant $\theta > 0$, $a + b \leq \max\left\{(1+\theta^{-1})a, (1+\theta)b\right\}$ for all $a, b \geq 0$, we obtain

$$\begin{aligned} \dot{W} &\leq -c(\phi^2 + \omega^2) + R^2 + (2c + \frac{31}{4})e_\phi^2 \\ &\leq -2cW + \max\left\{(1+\theta^{-1})R^2,\, (1+\theta)(2c + \frac{31}{4})e_\phi^2\right\}\,, \end{aligned} \tag{7.50}$$

from which it follows that

$$W \geq \max\left\{\frac{(1+\theta^{-1})}{1.9c}R^2,\, \frac{(1+\theta)}{1.9c}(2c + \frac{31}{4})e_\phi^2\right\} \quad \Rightarrow \quad \dot{W} \leq -0.1cW\,. \tag{7.51}$$

Then, (7.21) follows because $|\phi| \leq \sqrt{2W}$, and the gains are

$$\rho_{h\mu}(\cdot) = \sqrt{\frac{(1+\theta^{-1})}{0.95c}}\,(\cdot)\,, \quad \rho_{he}(\cdot) = \sqrt{(1+\theta)\left(\frac{2}{0.95} + \frac{31}{3.8c}\right)}\,(\cdot)\,. \tag{7.52}$$

Using (7.34), (7.41) and (7.52), the inner and outer loop small-gain conditions, (7.18) and (7.19) are, respectively,

$$2.1\sqrt{\frac{(1+\theta^{-1})}{0.95c}} \quad < \quad 1 \tag{7.53}$$

$$2.1\sqrt{\frac{6.1}{k}}\sqrt{(1+\theta)\left(\frac{2}{0.95} + \frac{31}{3.8c}\right)} \quad < \quad 1\,. \tag{7.54}$$

Selecting $c > 0$ and $k > 0$ sufficiently large ensures that (7.53) and (7.54) hold. Additional freedom for the selection of c and k is obtained from $\theta > 0$, which allocates the inner and outer loop gains.

7.4 Summary

We have discussed how the new observer developed in preceding chapters can be incorporated in output-feedback control design with control laws that guarantee boundedness in the presence of bounded observer errors. The combined use of the new observer design and small-gain control design tools has led to an output-feedback design procedure, illustrated on the jet engine compressor example. Such combined use of observer and controller design tools is a promising research direction for nonlinear output-feedback control.

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